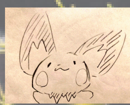
Astronomy and Astrophysics Olympiad (Theory and Data Analysis)

A Comprehensive Note Created by Former Hong Kong Team Members of IOAA

Team:

Lee Yiu Sing, Pika  (Leader of Theory and Data Analysis Note)

Chan Ngo Wang, Owen (Leader of Observation Note)

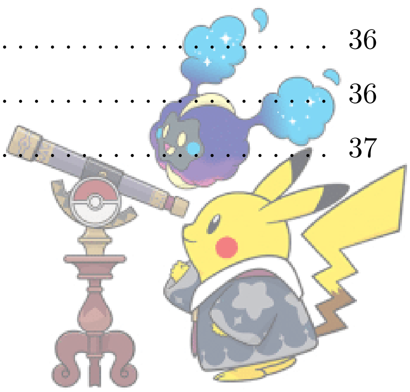
December 31, 2025

Contents

I Theory

1 Mathematical Physics 15

1.1 Basic Set Theory 15
1.2 Function 16
1.3 Trigonometry 16
1.4 Logarithm 18
1.5 Summation 18
1.6 Conic Section 19
1.7 Dimensional Analysis 19
1.7.1 Introduction 19
1.7.2 Buckingham Pi Theorem (Optional) 20
1.7.3 Natural Unit 21
1.7.4 Exercise 22
1.8 Tensor 23
1.8.1 Vector and Vector Space 23
1.8.2 Matrix 25
1.8.3 Tensor 28
1.8.4 Scalar Field and Vector Field 30
1.9 Differentiation and Partial Differentiation 33
1.9.1 Introduction 33
1.9.2 Properties and Formula 33
1.9.3 Partial Differentiation 34
1.9.4 Numerical Analysis 34
1.9.5 Linear Approximation 36
1.9.6 Taylor's Series 36
1.9.7 First Derivative Test 37



1.10 Integration . . . **39**

1.10.1 Primitive Function and Indefinite Integration . . . 39

1.10.2 Definite Integral . . . 40

1.10.3 Integration by Substitution . . . 42

1.10.4 Integration by Parts . . . 43

1.10.5 Multiple Integration, Line Integration and Surface Integration . . . 44

1.10.6 Trapezoidal Rule and Simpson's Rule . . . 45

1.10.7 Differential Equation . . . 46

1.11 Polar Coordinates . . . **47**

2 Spherical Trigonometry . . . **48**

2.1 Spherical Coordinates . . . **48**

2.2 Law of Sine and Law of Cosine . . . **48**

3 Astronomical Coordinate Systems . . . **50**

3.1 Horizontal Coordinate System . . . **50**

3.2 Equatorial Coordinate System . . . **51**

3.3 Ecliptic Coordinate System . . . **52**

3.3.1 Ecliptic Plane . . . 52

3.3.2 Obliquity of the Ecliptic . . . 52

3.3.3 Coordinate Definitions . . . 52

3.4 Galactic Coordinate System . . . **53**

3.4.1 Galactic Plane . . . 53

3.4.2 Galactic Center . . . 53

3.4.3 North Galactic Pole . . . 53

3.4.4 Coordinate Definitions . . . 53

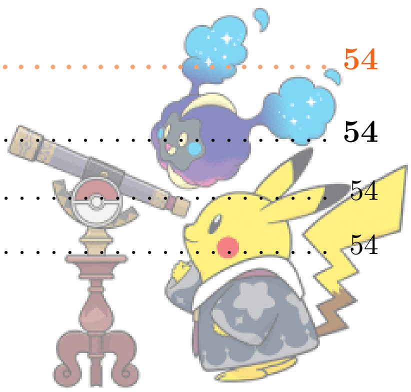
3.5 Circumpolar Stars . . . **54**

4 Astronomical Time . . . **54**

4.1 Solar Time . . . **54**

4.1.1 Definition . . . 54

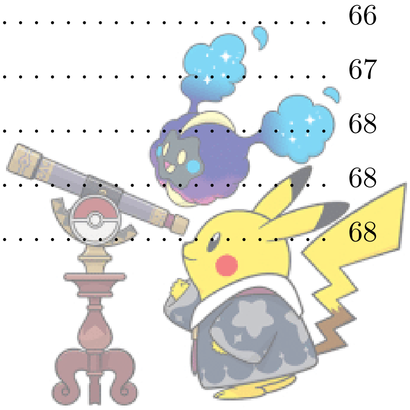
4.1.2 Apparent Solar Time . . . 54



4.1.3 Mean Solar Time 55
4.1.4 Equation of Time 55
4.2 Sidereal Time 55
4.2.1 Definition 55
4.2.2 Sidereal Day 55
4.3 Heliocentric Julian Date (HJD) 56
4.4 Local Mean Time (LMT) 56
4.5 Universal Time (UT) 56
4.6 Time Zones 57
4.7 Different Definitions of a Year 57

5 Basic Units in Astronomy 58

6 Fundamental Mechanics 58
6.1 Kinematics 58
6.1.1 Linear Motion 58
6.1.2 Relative Motion 59
6.2 Newton's Law 62
6.3 Linear Momentum 62
6.3.1 Introduction 62
6.3.2 Conservation of Linear Momentum 63
6.3.3 Impulse 63
6.3.4 Rocket Equation 63
6.4 Force, Torque and Equilibrium of Body 64
6.5 Energy 65
6.6 Circular Motion 66
6.6.1 Kinematics 66
6.6.2 Centripetal Force 67
6.6.3 Centrifugal Force 68
6.6.4 Coriolis Force 68
6.6.5 Angular Momentum 68



6.7 Case Study **70**

7 Orbital Mechanics 72

7.1 Newton's Law of Universal Gravitation **72**
7.1.1 Introduction 72
7.1.2 Barycentre 73
7.1.3 2-body Problem 74
7.1.4 n-body Problem 75
7.1.5 Binet's Equation 75

7.2 Kepler's Law **76**
7.2.1 Three Kepler's Laws 76
7.2.2 Kepler's Equation 82
7.2.3 Poisson Bracket 83

7.3 Shell Theorem **85**

7.4 Gravitational Potential Energy **87**
7.4.1 Vis-viva Equation 88
7.4.2 First Cosmic Speed, Second Cosmic Speed and Third Cosmic Speed 89
7.4.3 Hohmann Transfer Orbit 91
7.4.4 Blackhole 93
7.4.5 Virial Theorem 93
7.4.6 Tully-Fisher Relation 97

7.5 Lagrange Point **98**

7.6 Roche Limit **100**

8 Relativity 102

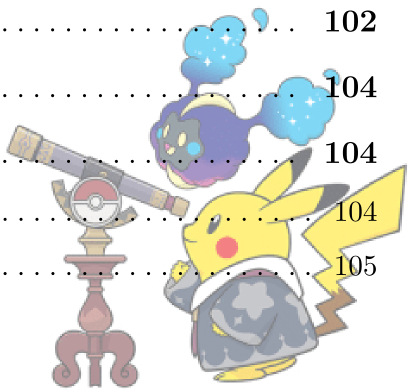
8.1 Postulates of Special Relativity **102**

8.2 Galileo Transformation **102**

8.3 Lorentz Transformation **102**

8.4 Spacetime Diagram **104**

8.5 Relativistic Kinematics and Mechanics **104**
8.5.1 Time Dilation 104
8.5.2 Length Contraction 105



8.5.3 Relativistic Momentum 105
8.5.4 Relativistic Energy 105
8.5.5 Relation between Relativistic Momentum and Energy 106

8.6 Spacetime and Four Vector 106

8.6.1 Spacetime in Special Relativity 106
8.6.2 Four-Vector 107
8.6.3 Metric Tensor 108
8.6.4 Invariant Interval 109
8.6.5 Important Four-Vectors in Physics 110

8.7 Introduction to General Relativity 112

8.7.1 Operators in General Relativity 112
8.7.2 Introduction to Spacetime and Differentiable Manifold 114
8.7.3 Metric Tensor 117
8.7.4 Geodesics 117
8.7.5 Riemann Curvature Tensor 117
8.7.6 Einstein's Field Equation 118
8.7.7 Solution to Einstein's Field Equation 118
8.7.8 Covariant Derivative 122
8.7.9 Lie Derivative 123
8.7.10 Killing Vector Field 124
8.7.11 Locally Measured Escape Speed in General Relativity 125

9 Thermodynamics 128

9.1 Kelvin 128

9.2 Pressure and Hydrostatic Equilibrium 128

9.3 Ideal Gas Law 130

9.4 First Law of Thermodynamics 130

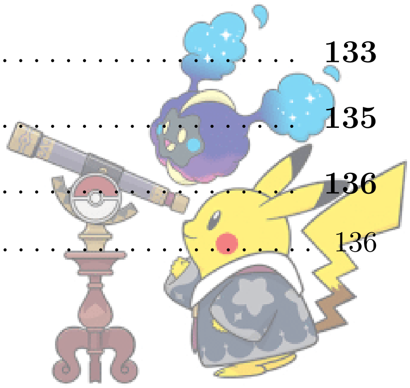
9.5 Second Law of Thermodynamics 132

9.6 Heat Capacity 133

9.7 Blackhole Thermodynamics 135

9.8 Kinetic Theory 136

9.8.1 Mean Free Path 136



9.8.2 Boltzmann Distribution and Maxwell–Boltzmann Distribution . . . 137

9.9 Boundary Conditions . . . 140

9.10 Case Study . . . 141

10 Spectroscopy . . . 144

10.1 Basic Concepts . . . 144

10.2 Spectral Radiance . . . 146

10.3 Stefan–Boltzmann law . . . 147

10.4 Doppler's Effect . . . 148

11 Electromagnetism . . . 150

11.1 Lorentz's Force . . . 150

11.2 Maxwell's Equation . . . 150

11.3 Poynting Vector . . . 151

11.4 Optics . . . 151

11.4.1 Wavefunction . . . 151

11.4.2 Lens . . . 152

11.5 Diffraction and Interference . . . 157

11.5.1 Principle of Superposition . . . 157

11.5.2 Complex Number . . . 158

11.5.3 Young's Double-Slit Experiment . . . 158

11.5.4 Single-Slit Diffraction . . . 159

11.5.5 Rayleigh's Criteria . . . 159

11.6 Polarization . . . 160

11.6.1 Introduction . . . 160

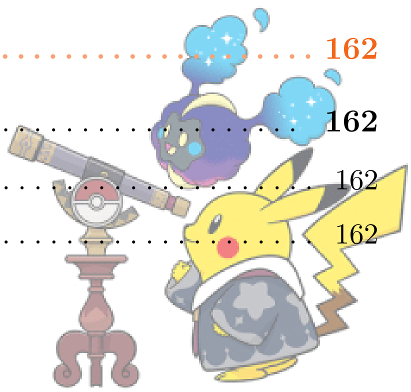
11.6.2 Faraday's Rotation . . . 161

12 Quantum Mechanics . . . 162

12.1 Atomic Structure . . . 162

12.1.1 Historical Development . . . 162

12.1.2 Introduction . . . 162



12.1.3 Nuclear Decay . . . 163
12.1.4 Neutrinos . . . 164
12.2 Wave-Particle Duality . . . 165
12.3 Planck's Equation . . . 165
12.4 Bohr Model of the Hydrogen Atom . . . 165
12.4.1 Bohr Postulates . . . 165
12.4.2 Formulas . . . 165
12.4.3 Wavefunction . . . 167
12.4.4 Time-Independent Schrödinger Equation . . . 167
12.5 Rydberg Formula . . . 169
12.6 Uncertainty Principle . . . 170
13 Stellar Astrophysics . . . 172
13.1 Stellar Classifications . . . 172
13.2 HR Diagram . . . 173
13.2.1 Introduction . . . 173
13.2.2 Spectral Types and Temperature . . . 174
13.2.3 Turn-Off Point . . . 176
13.3 Stellar Evolution . . . 176
13.3.1 Stellar Formation . . . 176
13.3.2 Pre-Main Sequence Stars . . . 177
13.3.3 Main Sequence Stars . . . 177
13.3.4 Post-Main Sequence Stars . . . 177
13.3.5 Supernovae . . . 178
13.3.6 Planetary Nebulae . . . 178
13.3.7 End States of Stars . . . 178
13.4 Magnitude System . . . 179
13.5 Albedo . . . 184
13.6 Geometric Albedo . . . 184
13.7 Color Indices . . . 185
13.8 Atmospheric Extinction . . . 186

13.9 Optical Depth 187

13.10 Binary Star 187

13.10.1 Different Types of Binary Stars 187

13.10.2 Modified Kepler's Third Law 188

13.10.3 Mass Function 188

13.10.4 Light Curves 190

13.10.5 Radial Velocity Curves 191

13.10.6 Roche Lobe 193

13.11 Exoplanet 194

13.11.1 Introduction 194

13.11.2 Classes of Exoplanets 194

13.11.3 Spectral Signatures of Possible Life 194

13.11.4 Radial Velocity Method 195

13.11.5 Transit Method 195

13.11.6 Habitable Zone 195

14 Cosmology 196

14.1 Structure of the Universe 196

14.1.1 Star Clusters 196

14.1.2 Galaxies 196

14.2 Large-scale Structure 199

14.3 Cosmological Principle 200

14.4 Rotational Curve 200

14.5 Hubble's Law 201

14.6 Cosmological Distance Measures 201

14.6.1 Proper Distance 201

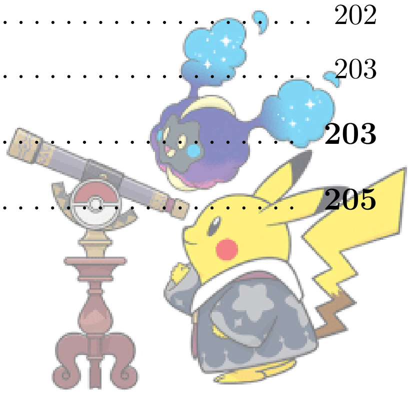
14.6.2 Comoving Distance 202

14.6.3 Luminosity Distance 202

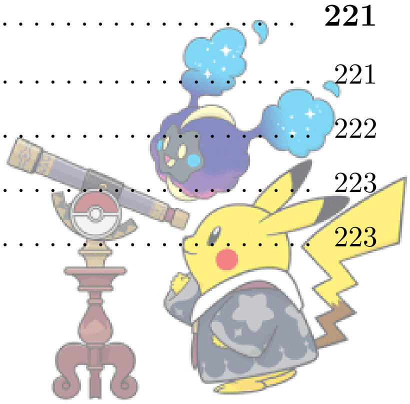
14.6.4 Angular Diameter Distance 203

14.7 Friedmann Equation 203

14.8 Equation of State 205



14.9 Big Bang **206**
14.9.1 Singularity 206
14.9.2 Cosmic Inflation 206
14.9.3 Expansion of Space 206
14.9.4 Phases of the Big Bang 206
14.10 Cosmic Microwave Background **208**
14.11 Gravitational Lensing **208**
14.11.1 Introduction 208
14.11.2 Derivation using Newtonian Mechanics 209
14.11.3 Derivation using General Relativity 211
14.12 Gravitational Wave **214**
14.12.1 Introduction 214
14.12.2 Chirp Mass 214
14.12.3 Binary System 214
14.13 Accretion Processes **214**
14.13.1 Introduction 214
14.13.2 Eddington Luminosity 215
14.14 Cosmic Distance Ladder **216**
14.14.1 Introduction 216
14.14.2 Radar Ranging 217
14.14.3 Stellar Parallax 218
14.14.4 Standard Candles: Cepheid Variables 218
14.14.5 Faber-Jackson Relations 219
14.14.6 Type Ia Supernovae 219
15 Interstellar Medium **221**
15.1 Introduction **221**
15.2 Fluid Dynamics **221**
15.2.1 Stress Tensor 221
15.2.2 Tensor Product 222
15.2.3 Divergence Theorem 223
15.2.4 Continuity Equation and Momentum Equation 223



15.2.5 Euler's Equation (Special Case of Navier-Stokes Equation) . . . 225

15.3 Case Study . . . **226**

15.3.1 Background . . . 226

15.3.2 Derivation . . . 226

16 Study of the Earth . . . **228**

16.1 Tides . . . **228**

16.2 Seasons . . . **228**

16.3 Factors Influencing Climate . . . **229**

16.4 Eclipses . . . **229**

16.4.1 Solar Eclipse . . . 229

16.4.2 Lunar Eclipse . . . 229

16.5 Space Weather . . . **229**

16.5.1 Introduction . . . 229

16.5.2 Solar Wind . . . 229

16.5.3 Solar Flares . . . 229

16.5.4 Coronal Mass Ejections (CMEs) . . . 230

16.5.5 Aurorae . . . 230

16.6 Meteor Showers . . . **230**

16.7 Equinoxes . . . **230**

16.8 Solstices . . . **230**

16.9 Solar Declination . . . **231**

17 Study of the Moon . . . **231**

17.1 Precession . . . **231**

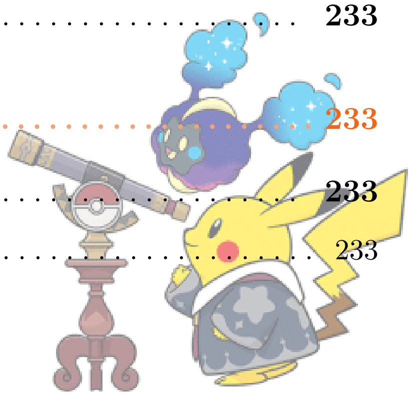
17.2 Nutation . . . **233**

17.3 Libration . . . **233**

18 Study of the Solar System . . . **233**

18.1 Formation . . . **233**

18.1.1 Nebular Hypothesis . . . 233



18.1.2 Differentiation and Evolution 234

18.2 Structure and Components of the Solar System 234

18.2.1 The Sun 234

18.2.2 Planets 234

18.2.3 Smaller Bodies 236

18.2.4 Kuiper Belt and Oort Cloud 236

19 Study of the Sun 236

19.1 Composition 236

19.2 Internal Structure 236

19.3 Atmosphere 237

19.4 Solar Surface Activities 237

19.4.1 Sunspots 237

19.4.2 Solar Wind 237

19.5 Magnetohydrodynamics 238

20 Human and Robotic Exploration within the Solar System 242

20.1 Human Exploration of the Solar System 242

20.2 Planetary Missions 243

II Data Analysis

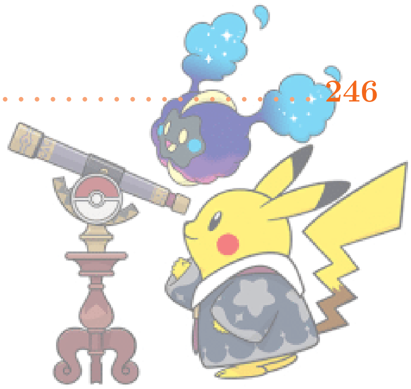
21 Probability 244

21.1 Introduction 244

21.2 Random Variables 244

22 Linear and Logarithmic Scale 245

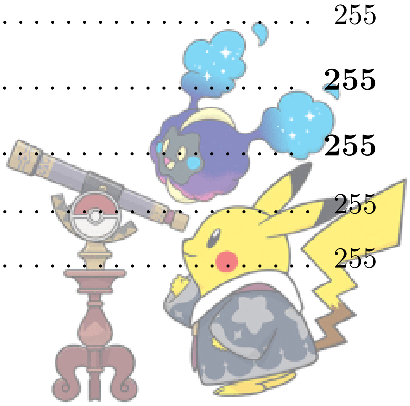
23 Measure of Central Tendancy 246



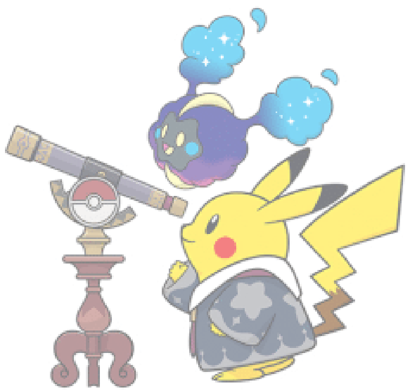
24 Measure of Dispersion 246
24.1 Basic Concepts 246
24.2 Box Plots (Box-and-Whisker) 247
25 Full Width at Half Maximum 247
26 Error Analysis 248
27 Regression Analysis 249
27.1 Linear regression 249
27.1.1 Introduction 249
27.1.2 Least Squares Method 249
27.2 Nonlinear Regression 250

III Observational Astronomy

28 Instrumentation and Space Technologies 251
28.1 Telescope 251
28.1.1 Type of Telescope 251
28.1.2 Mount Types 252
28.1.3 Key Components 252
28.1.4 Linear Magnification 252
28.1.5 Angular Magnification 252
28.1.6 Chromatic Aberration 253
28.1.7 f-number 253
28.1.8 Light-gathering Power 254
28.1.9 Adaptive Optics 254
28.1.10 Artificial Light 255
28.2 Interferometer 255
28.3 Detector 255
28.3.1 Photometers 255
28.3.2 Charge-Coupled Devices (CCDs) 255



28.4 Plate Scale **256**
28.4.1 Introduction 256
28.4.2 Field of View 257
28.5 Space-Based Instruments **257**
28.6 Signal-to-Noise Ratio **257**
28.6.1 Introduction 257
28.6.2 Pure Poisson Noise 258
28.6.3 Background Noise 258
28.6.4 Readout Noise 259
28.6.5 Complete Equation 259



Part I: Theory

1 Mathematical Physics

1.1 Basic Set Theory

A **set** is any collection of distinct objects, considered as an object in its own right. The objects in a set can be anything like real numbers $\mathbb{R}$. We denote a set by curly braces:

$$A = \{a_1, a_2, a_3, \dots\}$$

where $a_1, a_2, a_3, \dots$ are elements of the set A. We define several common operations on sets:

- **Union:** The union of two sets A and B is the set of all elements that are in either A or B (or both). Denoted as:
$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

- **Intersection:** The intersection of two sets A and B is the set of all elements that are in both A and B. Denoted as:
$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

- **Difference:** The difference of two sets A and B, denoted $A \setminus B$, is the set of all elements that are in A but not in B:
$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$$

- **Complement:** The complement of a set A, denoted A^c, is the set of all elements not in A.
$$A^c = \{x \mid x \notin A\}$$

We say that a set A is a **subset** of a set B, denoted $A \subseteq B$, if every element of A is also an element of B:

$$A \subseteq B \quad \text{if} \quad \forall x \, (x \in A \implies x \in B)$$

where $\forall$ means for all. Conversely, B is a **superset** of A, denoted $B \supseteq A$, if every element of A is in B.

1.2 Function

A **function** is a relationship between two sets, where each element of the first set (called the **domain**) is associated with exactly one element of the second set (called the **codomain**). A function assigns a unique output to every input from the domain. Formally, a function f from a set A to a set B is denoted as

$$f : A \to B$$

where A is the **domain** of the function, B is the **codomain** of the function. For each $a \in A$, there exists a unique $b \in B$ such that $f(a) = b$. There are several important types of functions that are commonly used in mathematics. Below are some of the most important ones:

- A function $f : A \to B$ is called **one-to-one** or **injective** if distinct elements in the domain A map to distinct elements in the codomain B. In other words:

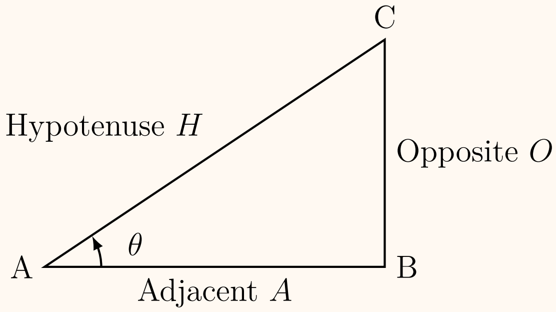
$$f(x_1) = f(x_2) \implies x_1 = x_2 \quad \text{for all} \quad x_1, x_2 \in A$$

- A function $f : A \to B$ is called **onto** or **surjective** if for every element $b \in B$, there exists at least one element $a \in A$ such that $f(a) = b$. In other words, the range of f is the entire codomain B.
- A function $f : A \to B$ is called **bijective** if it is both injective (one-to-one) and surjective (onto). This means that every element of the domain is mapped to a unique element in the codomain, and every element of the codomain has a corresponding element in the domain.

1.3 Trigonometry

Definition. 1.1: Trigonometric Function

Consider a right triangle with an angle θ, opposite side O, adjacent side A, and hypotenuse H. Recall that



For the angle θ in a right triangle:

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{O}{H}, \quad \csc\theta = \frac{H}{O}$$

$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{A}{H}, \quad \sec\theta = \frac{H}{A}$$

$$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{O}{A}, \quad \cot\theta = \frac{A}{O}$$

Definition. 1.2: Degree

Degrees are often divided into smaller units for more precise measurements. The most common smaller units are minutes and seconds:

- 1 degree (°) = 60 minutes (')
- 1 minute (') = 60 seconds (")

Definition. 1.3: Radian

A **radian** is defined as the angle subtended at the center of a circle by an arc whose length is equal to the radius of the circle:

$$\theta = \frac{s}{r}$$

where

- θ is the angle in radians,
- s is the length of the arc, and
- r is the radius of the circle.

Radians are closely related to degrees, but they are more natural for mathematical calculations. The relationship between radians and degrees is given by the following conversion formulas:

$$\theta_{\text{degrees}} = \theta_{\text{radians}} \times \frac{180}{\pi}$$

$$\theta_{\text{radians}} = \theta_{\text{degrees}} \times \frac{\pi}{180}$$

1.4 Logarithm

Recall that

$$\log_b x = y \iff b^y = x$$

Definition. 1.4: Natural Number e

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

which is $\left(1+\frac{1}{n}\right)^n$ when $n \to \infty$.

Define natural logarithm $\log_e x$ by ln. Note that

1. $\ln(xy) = \ln x + \ln y$
2. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$
3. $\ln(x^n) = n \ln x$

1.5 Summation

The summation symbol $\sum$ is used to denote the sum of a sequence of numbers:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

Einstein summation notation is a compact and powerful way to write sums over indexed quantities in tensor calculus. Instead of writing explicit summation signs, repeated indices in a term imply a sum over all values of that index. If an index appears twice in a single term (once as an upper index and once as a lower index, or in the same position in Euclidean space), it is implicitly summed over all its possible values:

$$A_i B_i \equiv \sum_{i=1}^{n} A_i B_i$$

Example. For two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, the dot product can be written using Einstein notation as:

$$\mathbf{u}\cdot\mathbf{v} = u_i v_i = \sum_{i=1}^{3} u_i v_i$$

1.6 Conic Section

A **conic section** is the curve obtained by intersecting a plane with a double-napped cone. The general form of the equation of a conic section in Cartesian coordinates is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where A, B, C, D, E, and F are constants that define the shape and orientation of the conic.

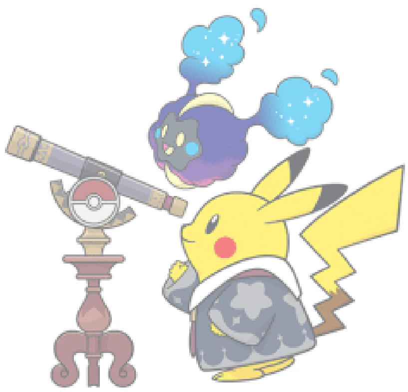
Conic Section	Equation and Properties
Circle	Equation: $(x-h)^2 + (y-k)^2 = r^2$ Center: (h, k) Radius: r
Ellipse	Equation: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ Center: (h, k) Semi-major axis: a, Semi-minor axis: b Orientation: If $a > b$, major axis along x-axis; if $b > a$, major axis along y-axis
Parabola	Equation: $y - k = a(x-h)^2$ Vertex: (h, k) Axis: Vertical
Hyperbola	Equation: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ Center: (h, k) Transverse axis along x-axis, Conjugate axis along y-axis

Table 1: Different Conic Sections and Their Properties

1.7 Dimensional Analysis

1.7.1 Introduction

Dimensional analysis is used to



- check the consistency of equations, and
- predict how different physical quantities relate to one another.

In essence, it is based on the idea that any physical quantity can be expressed in terms of a few fundamental dimensions, such as:

$$[M] \text{ (mass)}, \quad [L] \text{ (length)}, \quad [T] \text{ (time)}, \quad [I] \text{ (electric current)}, \quad [\Theta] \text{ (temperature)}$$

Every valid physical equation must be **dimensionally homogeneous**, meaning all additive terms must have the same dimensions.
Consider Newton's second law:

$$F = ma$$

The dimensions are:

$$[F] = [MLT^{-2}], \quad [m] = [M], \quad [a] = [LT^{-2}]$$

Since $[F] = [m][a]$, the equation is dimensionally consistent.

1.7.2 Buckingham Pi Theorem (Optional)

Given a physical relation described by:

$$f(Q_1, Q_2, \ldots, Q_n) = 0$$

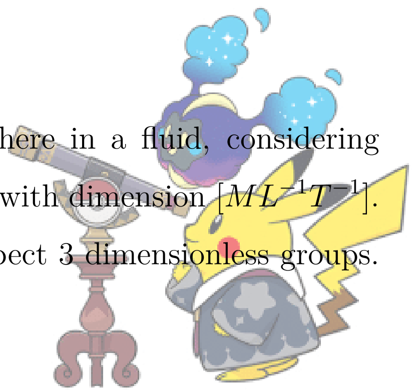
where Q_i are the n physical variables, and these n variables involve k independent fundamental dimensions, the equation can be rewritten as a function of $(n-k)$ independent dimensionless Π groups:

$$F(\Pi_1, \Pi_2, \ldots, \Pi_{n-k}) = 0$$

Each Π group is a dimensionless product of the original variables raised to some powers:

$$\Pi = Q_1^a Q_2^b Q_3^c \ldots$$

Example. Find the dimensionless relation for the drag force F_D on a sphere in a fluid, considering diameter D, fluid velocity V, fluid density ρ, g, and fluid dynamic viscosity μ with dimension $[ML^{-1}T^{-1}]$.
Solution. Note that $n = 6$ and $k = 3$ so number of pi terms $p = 3$ so we expect 3 dimensionless groups.



Choose Repeating Variables ($k = 3$): D, V and ρ (which are dimensionally independent and cover M, L and T). Then

$$\Pi_1 = F_D \cdot D^a V^b \rho^c$$

Solving,

$$\Pi_1 = \frac{F_D}{\rho V^2 D^2}$$

This is related to the drag coefficient C_D.

$$\Pi_2 = \mu \cdot D^a V^b \rho^c$$

Solving,

$$\Pi_2 = \frac{\mu}{\rho V D} = \mathrm{Re}$$

where Re is the Reynolds number.

$$\Pi_3 = g \cdot D^a V^b \rho^c$$

Solving,

$$\Pi_3 = \frac{gD}{V^2}$$

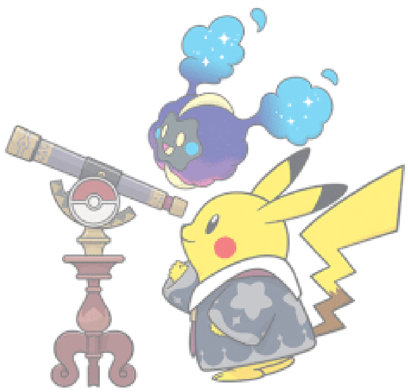
This is the inverse of the Froude number (Fr) squared.

$$F(C_D, \mathrm{Re}, \mathrm{Fr}) = 0 \Longrightarrow C_D = f(\mathrm{Re}, \mathrm{Fr})$$

1.7.3 Natural Unit

In physics, natural units are systems of units in which certain fundamental physical constants are set equal to 1. The most commonly used ones set the following fundamental constants to 1:

- $\hbar$ (The reduced Planck's constant)
- c (The speed of light in vacuum)
- G (The gravitational constant)
- k_B (Boltzmann constant)
- ϵ_0 (The permittivity of free space)



Dimensional analysis is used to restore the correct physical dimensions of quantities when transitioning between different unit systems.

Example. In natural units, the famous equation of mass-energy equivalence is given by

$$E = m$$

where E is energy and m is mass. However, in standard units, energy and mass have different dimensions. Energy is measured in joules (J) with the dimensions $[ML^2T^{-2}]$, while mass is measured in kilograms (kg) with the dimension $[M]$.

Using dimensional analysis, we restore the correct relationship by introducing the speed of light c. The energy and mass are related by the equation:

$$E = mc^2$$

Here, c (speed of light) has the dimension of $[LT^{-1}]$. Now the dimensions of both sides match because

$$[E] = [ML^2T^{-2}], \quad [mc^2] = [M][L^2T^{-2}]$$

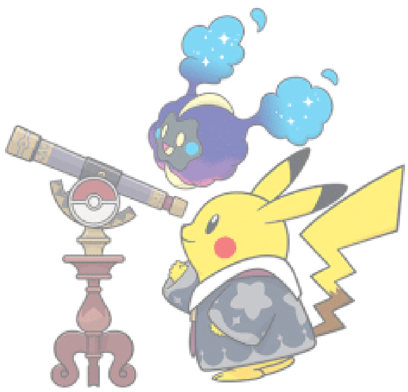
Hence, the restored equation in standard units is

$$E = mc^2$$

1.7.4 Exercise

The Planck unit system includes the following fundamental quantities: Planck Length (l_P), Planck Time (t_P), Planck Mass (m_P), Planck Charge (q_P), Planck Temperature (T_P), Planck Energy (E_P), Planck Force (F_P) and Planck Power (P_P). Each of these units is derived from combinations of the fundamental constants:

- The speed of light (c)
- The gravitational constant (G, with $[G] = [M^{-1}L^3T^{-2}]$)
- Planck’ s constant ($\hbar = h/2\pi$, with $[h] = [ML^2T^{-1}]$)
- Coulomb constant (ϵ_0. with $[\epsilon] = [ML^3T^{-4}I^{-2}]$)
- Boltzmann constant (k_B, with $[ML^2T^{-2}\Theta^{-1}]$)



Derive each of the quantities above.

1.8 Tensor

1.8.1 Vector and Vector Space

Definition. 1.5: Vector

A vector (denoted by an arrow above the letter ($\mathbf{A}$) or by a boldface letter ($\mathbf{A}$)) is a mathematical object that has both a magnitude (denoted by $|\mathbf{A}|$ or A) (size or length) and a direction. This is in contrast to a scalar, which only has magnitude (e.g., temperature, mass).

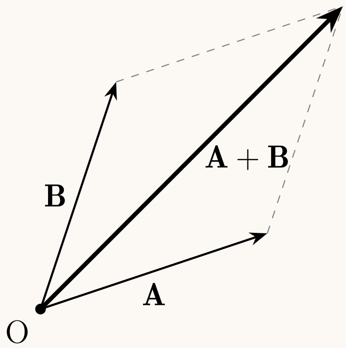
Definition. 1.6: Unit Vector

A unit vector ($\hat{u}$) has a magnitude of 1 and defines direction. The standard Cartesian unit vectors are $\mathbf{i}, \mathbf{j}, \mathbf{k}$ along the x, y, and z axes, respectively. A vector $\mathbf{A}$ in 3D space can be expressed in terms of its Cartesian components (A_x, A_y, A_z):

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}.$$

Theorem. 1.1: Tail-to-tail Method

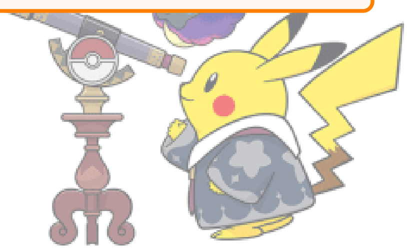
The sum of two vectors can be visualized as the diagonal of a parallelogram formed by the two vectors placed tail-to-tail:



Definition. 1.7: Dot Product

Given that angle between two vectors is θ. If $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$, then

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos(\theta) = A_xB_x + A_yB_y + A_zB_z$$



Theorem. 1.2: Projection

The projection of **a** onto **b** is given by the following formula:

$$\text{proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$$

Example. The effective area A_{eff} of a surface can be defined as the projection of the surface area in the direction of a particular vector. If **A** is a vector representing the surface area and $\hat{v}$ is a unit vector in the direction of interest, the effective area is given by the dot product:

$$A_{\text{eff}} = \mathbf{A} \cdot \hat{v}$$

where

- **A** is the vector representing the surface area (with magnitude representing the area and direction perpendicular to the surface),
- $\hat{v}$ is the unit vector in the direction of interest, and
- A_{eff} is the projection of the area in the direction of $\hat{v}$.

Definition. 1.8: Cross Product

Given that angle between two vectors is θ. If $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$, then the component form of cross product is given by

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

and the vector form of cross product is given by

$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}|\sin(\theta)\hat{n}$$

where $\hat{n}$ is a unit vector perpendicular to the plane formed by **a** and **b**, and its direction is given by the right-hand rule.

Example. Given two vectors $\mathbf{a}$ and $\mathbf{b}$, which form two sides of a triangle, the area A of the triangle is half the magnitude of the cross product of $\mathbf{a}$ and $\mathbf{b}$. Mathematically, the area is given by

$$A = \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$$

Definition. 1.9: Vector Space

A **vector space** V over a field $\mathbb{F}$ (e.g., $\mathbb{R}$ or $\mathbb{C}$) is a set of vectors equipped with two operations:

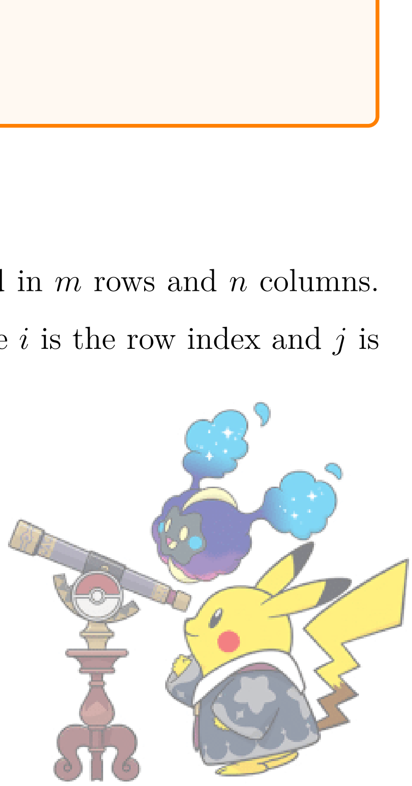
- **Vector addition:** $+ : V \times V \to V$
- **Scalar multiplication:** $\cdot : \mathbb{F} \times V \to V$

satisfying the following axioms for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and scalars $\alpha, \beta \in \mathbb{F}$:

- **Associativity of addition:** $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- **Commutativity of addition:** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- **Existence of zero vector:** There exists $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$
- **Existence of additive inverse:** For each $\mathbf{v} \in V$, there exists $-\mathbf{v} \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
- **Compatibility of scalar multiplication:** $\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$
- **Identity element of scalar multiplication:** $1 \cdot \mathbf{v} = \mathbf{v}$
- **Distributivity over vector addition:** $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$
- **Distributivity over scalar addition:** $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$

1.8.2 Matrix

Definition A **matrix** is an ordered rectangular array of numbers arranged in m rows and n columns. A matrix is typically denoted as A, and its elements are written as a_{ij}, where i is the row index and j is



the column index.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Here, A is an $m \times n$ matrix (with m rows and n columns).

Matrix Multiplication Matrix multiplication is defined as the dot product of rows of the first matrix with columns of the second matrix. If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then their product $C = AB$ is an $m \times p$ matrix, and its elements are given by

$$C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Scalar Multiplication A matrix A can be multiplied by a scalar λ (a number), which scales all the elements of the matrix:

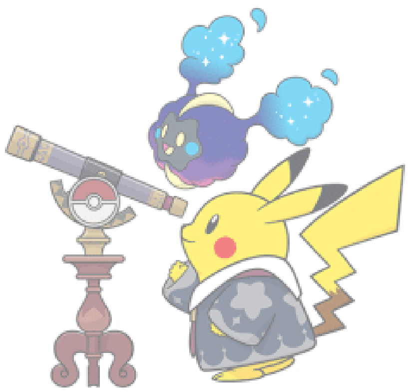
$$\lambda A = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1} & \lambda a_{m2} & \cdots & \lambda a_{mn} \end{pmatrix}$$

Determinant of a Matrix For a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the determinant is computed as

$$\det(A) = ad - bc$$



For larger matrices, the determinant can be computed using cofactor expansion. The determinant of an $n \times n$ matrix A is given by

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

where A_{ij} is the matrix obtained by removing the i-th row and j-th column of A, and a_{ij} is the element in the i-th row and j-th column.

Inverse of a Matrix

Definition. 1.10: Minor, Cofactor and Adjoint

Let $A = (a_{ij})$ be an $n \times n$ matrix. The **minor** M_{ij} of the entry a_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting the i-th row and j-th column of A.

The **cofactor** C_{ij} of a_{ij} is defined by

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Let A be an $n \times n$ matrix.

The **adjoint** (or **adjugate**) of A, denoted by $\operatorname{adj}(A)$, is given by

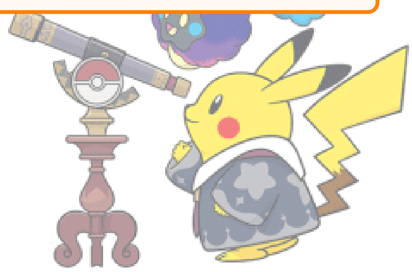
$$\operatorname{adj}(A) = (C_{ji})_{1 \le i,j \le n}$$

Definition. 1.11: Inverse Matrix

Let A be an $n \times n$ matrix. A matrix B is called the **inverse** of A if

$$AB = BA = I_n$$

where $I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$ is a $n \times n$ matrix. If such a matrix exists, A is called **invertible** and the inverse is denoted by A^{-1}.



Theorem. 1.3: Existence of the Inverse

An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$.

Theorem. 1.4: Inverse via the Adjoint

If A is an $n \times n$ matrix with $\det A \neq 0$, then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A)$$

1.8.3 Tensor

Basis For a vector space V, a set of vectors $\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ is a basis of V if

- **Linear Independence:** No vector in the set can be written as a linear combination of the others.
- **Spanning:** Every vector in V can be expressed as a linear combination of the basis vectors.

Example.

$$\mathcal{B} = \left\{ \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for $\mathbb{R}^2$ because the vectors are linearly independent, and any vector $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ can be written as

$$\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2$$

Change of Basis Let V be a vector space, and let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ be a basis for V. A vector $\mathbf{v} \in V$ can be written as a linear combination of the basis vectors:

$$\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$$

where $c_1, c_2, \dots, c_n$ are the coordinates of $\mathbf{v}$ with respect to the basis $\mathcal{B}$. These coordinates $\{c_1, c_2, \dots, c_n\}$ form the coordinate vector $[\mathbf{v}]_{\mathcal{B}}$ of $\mathbf{v}$ in the basis $\mathcal{B}$.

Consider two bases:

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$$

and

$$\mathcal{B}' = \{\mathbf{b}'_1, \mathbf{b}'_2, \ldots, \mathbf{b}'_n\}$$

Let $\mathbf{v}$ be a vector in V. The goal is to express the coordinate vector $[\mathbf{v}]_{\mathcal{B}}$ of $\mathbf{v}$ with respect to the new basis $\mathcal{B}'$. To do this, we need to find the relationship between the two coordinate systems. Suppose the new basis vectors $\mathbf{b}'_i$ are expressed as linear combinations of the old basis vectors $\mathbf{b}_i$:

$$\mathbf{b}'_i = \sum_{j=1}^{n} P_{ij}\mathbf{b}_j$$

where $P = [P_{ij}]$ is the change of basis matrix that transforms the old basis $\mathcal{B}$ to the new basis $\mathcal{B}'$.

Then, if $\mathbf{v}$ has coordinates $[\mathbf{v}]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ in the basis $\mathcal{B}$, its coordinates $[\mathbf{v}]_{\mathcal{B}'}$ in the new basis $\mathcal{B}'$ can be computed as:

$$[\mathbf{v}]_{\mathcal{B}'} = P^{-1}[\mathbf{v}]_{\mathcal{B}}$$

where P^{-1} is the inverse of the change of basis matrix.

Tensor A tensor is a mathematical object that generalizes scalars, vectors, and matrices. It can be thought of as a multi-dimensional array of numbers that transforms according to certain rules under a change of coordinates.

The order (or rank) of a tensor refers to the number of indices needed to describe it:

- A scalar is a tensor of order 0.
- A vector is a tensor of order 1.
- A matrix is a tensor of order 2.
- A general tensor may have order 3, 4, or higher, depending on the number of indices.

The components of a tensor are typically denoted with indices. For example, a second-order tensor T with components T_{ij} can be written as:

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

Contravariant Tensor A contravariant tensor has components that transform in a certain way when we change the coordinates. Specifically, the components of a contravariant tensor transform according to the inverse of the transformation matrix. Let T be a contravariant vector. When we change coordinates, the components T^i transform as:

$$T'^i = \sum_j A^i_j T^j$$

where A^i_j is the transformation matrix for the new coordinates. In index notation, the upper index (superscript) indicates that the tensor is contravariant.

For example, consider a contravariant vector $\mathbf{v} = (v^1, v^2)$. Under a change of coordinates, the components transform as

$$v'^1 = A^1_1 v^1 + A^1_2 v^2, \quad v'^2 = A^2_1 v^1 + A^2_2 v^2$$

Covariant Tensor In contrast, a covariant tensor has components that transform differently. The components of a covariant tensor transform according to the transformation matrix itself. Let S be a covariant vector. Under a change of coordinates, the components S_i transform as

$$S'_i = \sum_j A^j_i S_j$$

where A^j_i is the transformation matrix for the new coordinates. In index notation, the lower index (subscript) indicates that the tensor is covariant. For example, consider a covariant vector $\mathbf{w} = (w_1, w_2)$. Under a change of coordinates, the components transform as

$$w'_1 = A^1_1 w_1 + A^2_1 w_2, \quad w'_2 = A^1_2 w_1 + A^2_2 w_2$$

1.8.4 Scalar Field and Vector Field

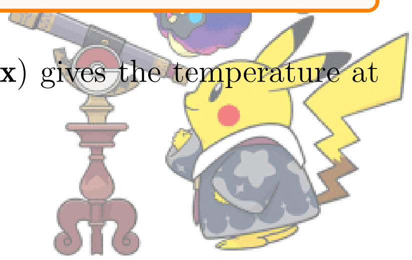
Definition. 1.12: Scalar Field

A **scalar field** is a mathematical function that assigns a scalar value to every point in a space. It is represented as

$$\phi : \mathbb{R}^n \to \mathbb{R}, \quad \mathbf{x} \mapsto \phi(\mathbf{x})$$

where $\mathbb{R}^n$ is the space in which the field is defined, and $\phi(\mathbf{x})$ is a scalar value at the point $\mathbf{x}$.

Example. The temperature at every point in a room is a scalar field. If $T(\mathbf{x})$ gives the temperature at point $\mathbf{x}$, then T is a scalar field.



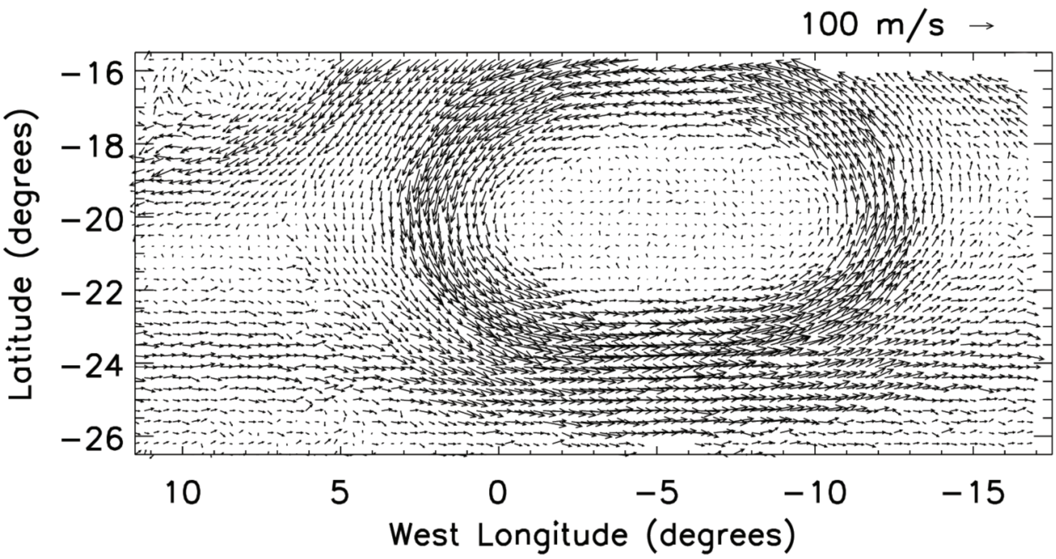
Definition. 1.13: Vector Field

A **vector field** is a function that assigns a vector to every point in a space. It is represented as

$$\mathbf{V} : \mathbb{R}^n \to \mathbb{R}^n, \quad \mathbf{x} \mapsto \mathbf{V}(\mathbf{x})$$

where $\mathbf{V}(\mathbf{x})$ is the vector field at point $\mathbf{x}$, and $\mathbb{R}^n$ is the space in which the vector field is defined.

Exercise. (2020 GeCAA)



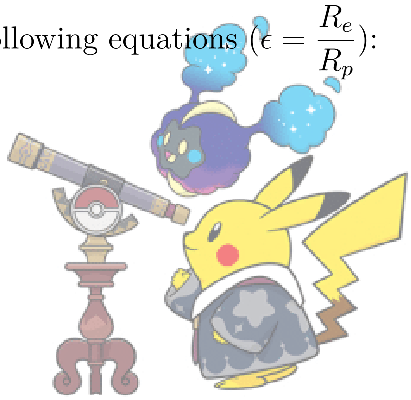
In the following problem, the fluid mechanics of Jupiter's Great Red Spot (GRS) is studied based on the velocity field data. The diagram above shows a map of relative velocity for GRS and the surrounding region. The arrows are oriented and scaled as per the directions and magnitudes of winds at different points. Due to the combined effects of gravity and rotation, Jupiter is slightly flattened at its poles. The equation of a spheroid approximating for the shape of Jupiter can be stated as:

$$\frac{x^2}{R_e^2} + \frac{y^2}{R_p^2} = 1$$

where $R_e = 7.15 \times 10^7\, m$ is the equatorial radius of Jupiter, and $R_p = 6.69 \times 10^7\, m$ the polar radius. The radii of curvature of this spheroid in any direction can be calculated by the following equations ($\epsilon = \frac{R_e}{R_p}$):

$$r(\phi) = R_e(1 + \epsilon^2 \tan^2 \phi)^{1/2}$$

$$R(\phi) = R_e \epsilon^2 \left(\frac{r(\phi)}{R_e \cos \phi} \right)^3$$



where $r(\phi)$ and $R(\phi)$ are the zonal (aka the zone of a particular latitude) and meridional (aka longitudinal) radii of curvature, respectively, as a function of planetographic latitude ϕ. The sidereal rotation period of Jupiter is $P = 3.57 \times 10^4\, s$.

(a) Calculate the zonal and meridional radii values ($\bar{r}$ and $\bar{R}$) respectively at the location of the centre of the GRS.

(b) Estimate the eccentricity of the GRS.

(c) 'Vorticity' at any point is a measure of local spinning of the fluid as measured by an observer situated in the reference frame of the fluid. Mathematically, it is calculated as 'curl' (vector derivative product) of the velocity field. In this case, the average relative vorticity may be estimated by the equation:

$$\zeta = \frac{V_v}{L_{GRS} A_{GRS}}$$

where V_v is the maximum value of winds as per the velocity field, L_{GRS} is the length of the circumference of the GRS and A_{GRS} is the area of the GRS. Estimate average relative vorticity of the GRS.
Hint: The circumference of an ellipse is well approximated by an average of circumferences of the corresponding auxiliary and minor circles.

(d) Find the absolute vorticity $\zeta_a = (\zeta + f)$ by adding the Coriolis parameter:

$$f = 2\Omega \sin\phi$$

where Ω is the angular velocity of Jupiter (due to axial rotation) and ϕ is the appropriate latitude.

(e) If the absolute vorticity has the same sign as the latitude, we call the storm a 'cyclonic storm'. If they have opposite signs, the system is 'anticyclonic'. Is the GRS cyclonic or anticyclonic?

(f) Imagine that the GRS moves to another latitude ϕ_1, where the absolute vorticity changes the sign (changes from anti-cyclonic to cyclonic or vice versa). Assuming minimum possible displacement of the GRS, at what value of ϕ_1 do we expect this change?
In your analysis, assume that the GRS at the new location would occupy the same angular span in latitude, as well as have the same wind velocities and eccentricity as the original.

1.9 Differentiation and Partial Differentiation

1.9.1 Introduction

Differentiation is used for functions of a single independent variable, $y = f(x)$. It measures the instaneous rate of change of y with respect to x. The derivative of a function $f(x)$ with respect to x is defined using a limit and is denoted by $\frac{df}{dx}$ or $f'(x)$.

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \text{ (First principle)}$$

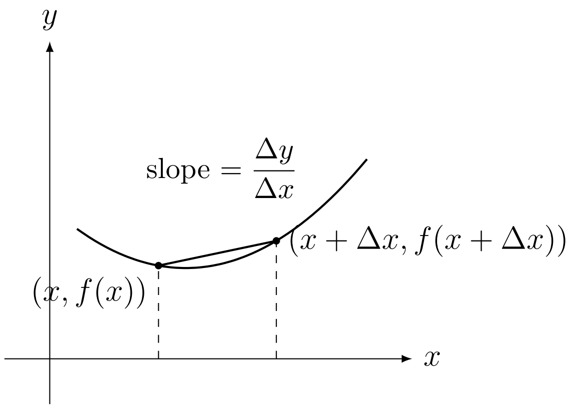


Figure 1: Differentiation is the slope when $\Delta x \to 0$

1.9.2 Properties and Formula

Definition. 1.14: Basic Properties

If $f(x)$ and $g(x)$ are differentiable functions (the derivative exists), c and n are any real numbers,

$(cf)' = cf'(x)$

$(f \pm g)' = f'(x) \pm g'(x)$

$(fg)' = f'g + fg'$ **(Product Rule)**

$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ **(Quotient Rule)**

$\frac{d}{dx}(c) = 0$

$\frac{d}{dx}(x^n) = nx^{n-1}$ **(Power Rule)**

$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$ **(Chain Rule)**

Definition. 1.15: Formulas

$\frac{d}{dx}(x) = 1$	$\frac{d}{dx}(\csc x) = -\csc x \cot x$	$\frac{d}{dx}(a^x) = a^x \ln(a),\ a > 0$
$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\cot x) = -\csc^2 x$	$\frac{d}{dx}(e^x) = e^x$
$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\ln(x)) = \frac{1}{x},\ x > 0$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\ln \lvert x \rvert) = \frac{1}{x},\ x \neq 0$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\frac{d}{dx}(\log_a(x)) = \frac{1}{x \ln a},\ x > 0$

1.9.3 Partial Differentiation

Partial differentiation is used for functions of **two or more independent variables**, such as $z = f(x, y)$. A partial derivative measures the rate of change of the function with respect to **one** variable, while **holding all other variables constant**. For example, $\frac{\partial z}{\partial x}$ is the derivative with respect to x while keeping y constant.

Example. Consider the function

$$f(x, y) = x^2 y + e^{xy}$$

$$\frac{\partial}{\partial x}\left(x^2 y + e^{xy}\right) = 2xy + ye^{xy}$$

1.9.4 Numerical Analysis

Definition. 1.16: Bisection Method

Let the function $f(x)$ be continuous on the interval $[a, b]$, and suppose that $f(a)$ and $f(b)$ have opposite signs, i.e.,

$$f(a) \cdot f(b) < 0$$

The root r lies between a and b, and we can approximate the root by iteratively halving the interval. The midpoint m of the interval is given by

$$m = \frac{a+b}{2}$$

We then check the sign of $f(m)$:

- If $f(m) = 0$, then m is the root.
- If $f(a) \cdot f(m) < 0$, the root lies between a and m, so we set $b = m$.
- If $f(m) \cdot f(b) < 0$, the root lies between m and b, so we set $a = m$.

This process is repeated until the interval becomes sufficiently small.

Example. To find the root of $f(x) = x^2 - 2$ on the interval $[1, 2]$:

- $f(1) = 1^2 - 2 = -1$
- $f(2) = 2^2 - 2 = 2$
- Since $f(1) \cdot f(2) < 0$, the root lies in $[1, 2]$.
- First approximation: $m_1 = \dfrac{1+2}{2} = 1.5$.
- $f(1.5) = 1.5^2 - 2 = 0.25$. Since $f(1) \cdot f(1.5) < 0$, the new interval is $[1, 1.5]$.

Definition. 1.17: Newton–Raphson method

Let $f(x)$ be a differentiable function, and let x_0 be an initial guess for the root. The next approximation x_1 is given by

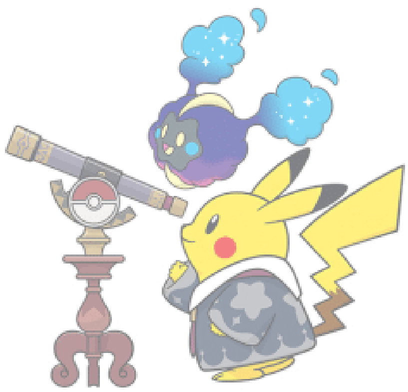
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

The method continues iteratively:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example. To solve $f(x) = x^2 - 2 = 0$, we find the derivative $f'(x) = 2x$. Using an initial guess $x_0 = 1.5$:

$$x_1 = x_0 - \frac{x_0^2 - 2}{2x_0}$$
$$x_1 = 1.5 - \frac{1.5^2 - 2}{2(1.5)} \approx 1.4167$$



1.9.5 Linear Approximation

To find the linear approximation, we begin by recalling the definition of the tangent line to a curve at a point. The equation of the tangent line at $x = a$ is

$$y = f'(a)(x - a) + f(a)$$

which provides the best linear approximation to the function near $x = a$:

$$f(x) \approx f(a) + f'(a)(x - a)$$

Example. $\sin x \approx x$ for small x in radian.

1.9.6 Taylor's Series

The Taylor Series of a function $f(x)$ about a point a is given by

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f^{(3)}(a)}{3!}(x - a)^3 + \cdots$$

where $f^{(n)}(x)$ is the n-th derivative of $f(x)$.
Example. Consider the function $f(x) = e^x$. The derivatives of e^x are

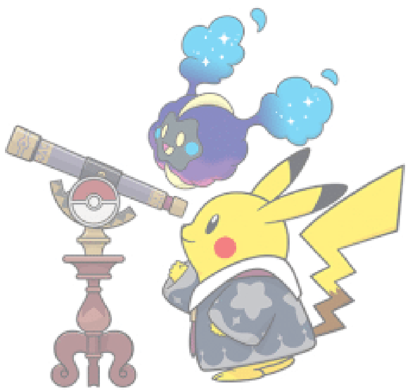
$$f'(x) = e^x, \quad f''(x) = e^x, \quad f^{(3)}(x) = e^x, \quad \ldots$$

Evaluating these at $x = 0$, we get

$$f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 1, \quad f^{(n)}(0) = 1 \text{ for all } n$$

Hence, the Taylor series of e^x around $x = 0$ (also known as the Maclaurin series) is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$



1.9.7 First Derivative Test

Definition. 1.18: Critical Point

Let $f(x)$ be a function that is differentiable on an interval containing c, except possibly at c itself.

Critical points of f are points c where either $f'(c) = 0$ or $f'(c)$ does not exist.

Theorem. 1.5: First Derivative Test

Suppose c is a critical point of a function f.

- If $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, then f has a **local maximum** at c.
- If $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, then f has a **local minimum** at c.
- If $f'(x)$ has the same sign on both sides of c, then f has **no local extremum** at c.

Example. (2025 IOAA) The Event Horizon Telescope (EHT) has released an image of the supermassive black hole at the centre of the M87 galaxy, as shown in the left panel of Fig. 2. To understand some simple features of this image, we consider a simplified model of a non-rotating, static, spherically symmetric black hole of mass

$$M = 6.5 \times 10^9 M_\odot$$

surrounded by a massless, thin, planar accretion disk of inner and outer radii

$$a_{\text{inner}} = 6R_{\text{SC}}, \qquad a_{\text{outer}} = 10R_{\text{SC}}$$

respectively, where R_{SC} is the Schwarzschild radius. A face-on view sketch of this system is shown in the right panel of Fig. 2. (The figure is not to scale.)

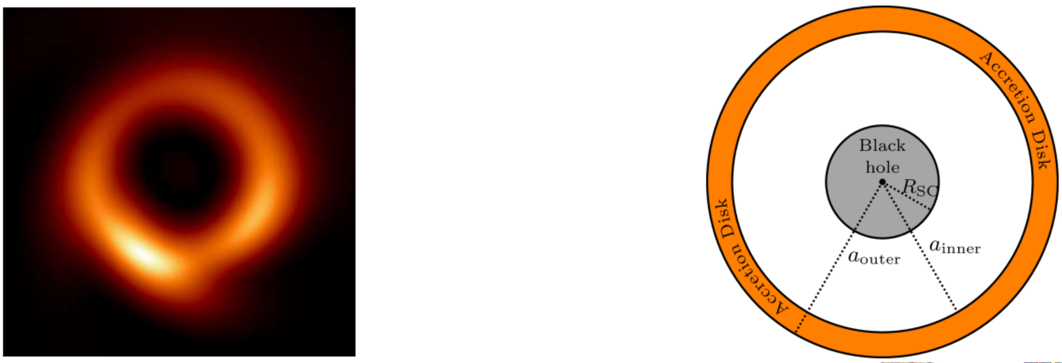


Figure 2: Left: EHT image of the black hole at the centre of M87. Right: Face-on schematic diagram of a black hole surrounded by a thin accretion disk.

We assume that the accretion disk is the only source of light to be considered. Every point on the disk emits light in all directions. This light travels under the influence of the gravitational field of the black hole. The path of the light rays is governed by the following two equations (which are similar to those describing the motion of an object around the Sun):

$$\frac{1}{2}v_r^2 + \frac{L^2}{2r^2}\left(1 - \frac{2GM}{c^2 r}\right) = E \tag{1}$$

$$v_\phi = r\omega = \frac{L}{r} \tag{2}$$

where $r \in (R_{\text{SC}}, \infty)$ is the radial coordinate, $\phi \in [0, 2\pi)$ is the azimuthal angle, and E and L are constants related to the conserved energy and conserved angular momentum, respectively. Here, $v_r = \frac{dr}{dt}$ is the magnitude of the radial velocity, v_ϕ is the magnitude of the tangential velocity, and $\omega = \frac{d\phi}{dt}$ is the angular velocity. We define the impact parameter b for a trajectory as

$$b = \frac{L}{\sqrt{2E}}$$

Time dilation effects are neglected in this problem. Another useful equation is obtained by differentiating the first equation with respect to time:

$$\frac{dv_r}{dt} - \frac{L^2}{r^3} + \frac{3GML^2}{c^2 r^4} = 0 \tag{3}$$

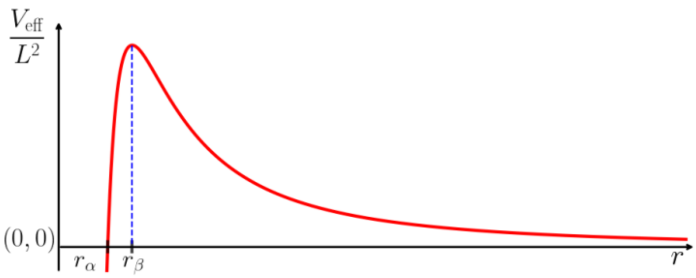
(a) Circular light trajectories can exist around the black hole. Find the radius r_{ph} and the impact parameter b_{ph} for such photon trajectories in terms of M and relevant constants.

(b) Calculate the time T_{ph} taken to complete one full orbit of the circular light trajectory, in seconds.

(c) The radial velocity equation for light trajectories can be compared with an equation of the form

$$\frac{v_r^2}{2} + V_{\text{eff}}(r) = E \tag{4}$$

A schematic plot of V_{eff}/L^2 as a function of r is shown in the following figure. The plot indicates two special radii, r_α and r_β. Obtain expressions for r_α and r_β in terms of M and relevant constants.



1.10 Integration

1.10.1 Primitive Function and Indefinite Integration

The **primitive function** (or antiderivative) of a given function $f(x)$ is a function $F(x)$ such that

$$F'(x) = f(x)$$

Example. Let $f(x) = 2x$. The corresponding primitive function $F(x)$ is

$$F(x) = x^2 + C$$

where C is a constant, called the constant of integration. This constant appears because the derivative of any constant is zero, and thus there are infinitely many functions that differ only by a constant.

The **indefinite integral** of a function $f(x)$ is the operation of finding its primitive function. It is written as

$$\int f(x)\,dx = F(x) + C$$

Some important properties of indefinite integrals include

Theorem. 1.6: Properties

- **Linearity:**
$$\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$
- **Constant factor:**
$$\int cf(x)\,dx = c\int f(x)\,dx$$
where c is a constant.

Theorem. 1.7: Basic Primitive Function

- **Power Rule:**

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad \text{for} \quad n \neq -1$$

- **Exponential Functions:**

$$\int e^x \, dx = e^x + C$$

- **Trigonometric Functions:**

$$\int \sin(x) \, dx = -\cos(x) + C$$

$$\int \cos(x) \, dx = \sin(x) + C$$

1.10.2 Definite Integral

Definition. 1.19: Riemann Sum

Let $f(x)$ be a function defined on the interval $[a, b]$, and let the interval be divided into n subintervals:

$$[a, b] = [x_0, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_{n-1}, x_n]$$

where $x_0 = a$ and $x_n = b$, and the partition points

$$x_0, x_1, \ldots, x_n$$

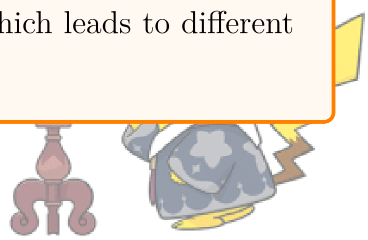
are chosen such that $a = x_0 < x_1 < \cdots < x_n = b$. The Riemann sum is defined as

$$S_n = \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

where:

- x_i^* is a sample point chosen from the i-th subinterval $[x_{i-1}, x_i]$,
- $\Delta x_i = x_i - x_{i-1}$ is the width of the i-th subinterval.

There are several ways to choose the sample points x_i^* in each subinterval, which leads to different types of Riemann sums:



- **Left Riemann Sum:** $x_i^* = x_{i-1}$, the left endpoint of each subinterval.
- **Right Riemann Sum:** $x_i^* = x_i$, the right endpoint of each subinterval.
- **Midpoint Riemann Sum:** $x_i^* = \dfrac{x_{i-1} + x_i}{2}$, the midpoint of each subinterval.

Definition. 1.20: Definite Integral

The definite integral is defined as the limit of the Riemann sum as the number of subintervals $n \to \infty$ and the maximum subinterval width $\Delta x_i \to 0$:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

The definite integral can be interpreted as the signed area under the curve $y = f(x)$ from $x = a$ to $x = b$. If $f(x)$ is above the x-axis, the area is positive, and if $f(x)$ is below the x-axis, the area is negative.

Theorem. 1.8: Fundamental Theorem of Calculus

Alternatively, the definite integral can be computed using the Fundamental Theorem of Calculus:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is any antiderivative of $f(x)$.

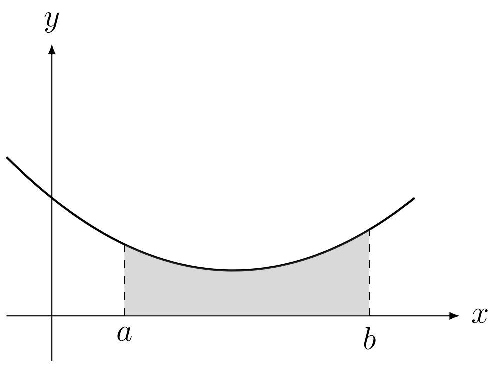


Figure 3: Signed area under the curve $y = f(x)$ from $x = a$ to $x = b$ is given by $\int_a^b f(x)\,dx$.

Some important properties of definite integrals include

Definition. 1.21: Properties

- **Linearity:**
$$\int_a^b (c_1 f(x) + c_2 g(x))\, dx = c_1 \int_a^b f(x)\, dx + c_2 \int_a^b g(x)\, dx$$
where c_1 and c_2 are constants.

- **Additivity:**
$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$
for any point c between a and b.

- **Reversal of Limits:**
$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$
Reversing the limits of integration changes the sign of the integral.

- **Zero Width Integral:**
$$\int_a^a f(x)\, dx = 0$$
The integral over a single point is always zero.

1.10.3 Integration by Substitution

Theorem. 1.9: Integration by Substitution in Indefinite Integral

Suppose we have the following indefinite integral:

$$\int f(g(x)) \cdot g'(x)\, dx$$

We can use substitution by letting $u = g(x)$, so that $du = g'(x)\, dx$. This changes the integral into

$$\int f(u)\, du$$

Now, we can integrate with respect to u and then substitute back in terms of x.

Example. Consider the integral

$$\int x e^{x^2}\, dx$$

Let

$$u = x^2, \quad du = 2x\,dx$$

Hence, the integral becomes

$$\frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C$$

Theorem. 1.10: Integration by Substitution in Definite Integral

For definite integrals, the limits of integration change according to the substitution. Suppose we have

$$\int_a^b f(g(x)) \cdot g'(x)\,dx$$

Let $u = g(x)$, then the limits of integration transform as follows:

$$u(a) = g(a) \quad \text{and} \quad u(b) = g(b)$$

The integral becomes

$$\int_{g(a)}^{g(b)} f(u)\,du$$

We can now evaluate the definite integral in terms of u.

1.10.4 Integration by Parts

Theorem. 1.11: Integration by Part in Definite Integral

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx$$

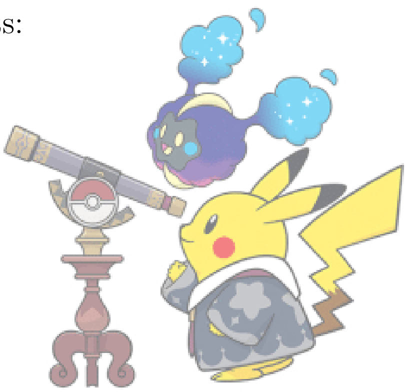
where $u(x)$ is chosen to simplify $u'(x)$ and $v'(x)$ is easily integrable.

Proof. Integration by parts is based on the product rule for differentiation:

$$\frac{d}{dx}[u(x)v(x)] = u'(x)v(x) + u(x)v'(x)$$

Rewriting and integrating both sides gives the formula for integration by parts:

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx$$



Example. Evaluate the integral

$$\int xe^x \, dx$$

Let $u = x$, so that $du = dx$, and let $dv = e^x \, dx$, so that $v = e^x$. Using the integration by parts formula:

$$\int xe^x \, dx = xe^x - \int e^x \, dx = xe^x - e^x + C$$

Theorem. 1.12: Integration by Part in Indefinite Integral

To solve an integral using integration by parts, we choose u and dv from the integrand. Differentiating u gives du, and integrating dv gives v. Substituting into the formula, we obtain:

$$\int u \, dv = uv - \int v \, du$$

1.10.5 Multiple Integration, Line Integration and Surface Integration

Definition. 1.22: n-dimensional Integration

In n dimensions, the integral of a function $f : \mathbb{R}^n \to \mathbb{R}$ over a region $D \subset \mathbb{R}^n$ is given by

$$\int_D f(\mathbf{x}) \, d\mathbf{x} = \int_{x_1=a_1}^{b_1} \int_{x_2=a_2}^{b_2} \cdots \int_{x_n=a_n}^{b_n} f(x_1, x_2, \ldots, x_n) \, dx_n \ldots dx_2 \, dx_1$$

Example. the 2-dimensional integral of a function $f(x, y)$ over a rectangle $[a_1, b_1] \times [a_2, b_2]$ is

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} f(x, y) \, dy \, dx$$

Definition. 1.23: Line Integration

A line integral is an integral of a function along a curve or path. Let $\mathbf{r}(t) = (x(t), y(t), z(t))$ be a parametrization of the curve C, with t ranging from a to b. Then, the line integral of a scalar function f along C is given by

$$\int_C f(x, y, z) \, ds = \int_a^b f(x(t), y(t), z(t)) \left| \frac{d\mathbf{r}}{dt} \right| dt$$

If the integrand involves a vector field $\mathbf{F} = (F_1, F_2, F_3)$, the line integral becomes

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \left(F_1(x(t), y(t), z(t)) \frac{dx}{dt} + F_2(x(t), y(t), z(t)) \frac{dy}{dt} + F_3(x(t), y(t), z(t)) \frac{dz}{dt} \right) dt$$

Definition. 1.24: Surface Integration

A surface integral involves integrating over a surface S. The surface integral of a scalar function $f(x, y, z)$ over a surface S is

$$\int_S f(x, y, z)\, dS = \iint_D f(x(u, v), y(u, v), z(u, v)) \left| \frac{\partial(x, y, z)}{\partial(u, v)} \right| du\, dv$$

where $(x(u, v), y(u, v), z(u, v))$ is the parametrization of the surface, and the determinant term is the Jacobian of the transformation from the (u, v) parameter space to the (x, y, z) space.

For a vector field $\mathbf{F} = (F_1, F_2, F_3)$, the surface integral becomes

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(x(u, v), y(u, v), z(u, v)) \cdot \mathbf{n}(u, v) \left| \frac{\partial(x, y, z)}{\partial(u, v)} \right| du\, dv$$

where $\mathbf{n}(u, v)$ is the unit normal vector to the surface at each point.

1.10.6 Trapezoidal Rule and Simpson's Rule

For more complex functions where integration is difficult, numerical methods such as the Trapezoidal Rule can be used.

Theorem. 1.13: Trapezoidal Rule

Let f be a continuous function on $[a, b]$. Partition the interval into n equal subintervals of width

$$h = \frac{b - a}{n}$$

and define the partition points by $x_i = a + ih$ for $i = 0, 1, \ldots, n$.

The Trapezoidal Rule approximates the definite integral

$$\int_a^b f(x)\, dx$$

by

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right]$$

This method corresponds to approximating the graph of f on each subinterval by a straight line and summing the areas of the resulting trapezoids.

Another more accurate numerical approximation is Simpson's Rule, which approximates the area under the curve by fitting parabolas to segments of the curve.

Theorem. 1.14: Simpson's Rule

Let f be a continuous function on $[a, b]$, and suppose that n is an **even** positive integer. With $h = \dfrac{b-a}{n}$ and partition points $x_i = a + ih$, Simpson's Rule approximates

$$\int_a^b f(x)\,dx$$

by

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[f(a) + 4\sum_{\substack{i=1 \\ i \text{ odd}}}^{n-1} f(x_i) + 2\sum_{\substack{i=2 \\ i \text{ even}}}^{n-2} f(x_i) + f(b)\right]$$

This method is derived by approximating f on each pair of subintervals using a parabola, resulting in significantly higher accuracy for sufficiently smooth functions.

1.10.7 Differential Equation

Introduction An **ordinary differential equation** (ODE) is an equation that involves a function and its derivatives. A general first-order ODE has the form

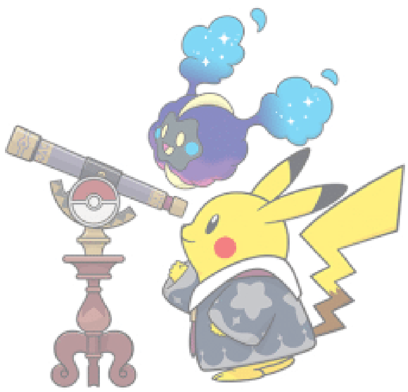
$$\frac{dy}{dx} = f(x, y)$$

Separable ODE A first-order ODE is **separable** if it can be written as

$$\frac{dy}{dx} = g(x)h(y)$$

Then we solve by separating variables:

$$\frac{1}{h(y)}dy = g(x)dx$$



and integrate both sides.

Integrating Factor A first-order linear ODE has the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

can be solved by introducing

$$\mu(x) = e^{\int P(x)dx}$$

$$y\mu(x) = \int Q(x)\mu(x)dx$$

Initial Condition An **initial condition** is a specification of the value of the solution (and possibly its derivatives) at a particular point, usually denoted t_0 or x_0. For an n-th order ODE, we typically need n initial conditions to determine a unique solution.

1.11 Polar Coordinates

A point P in the plane is described in polar coordinates by an ordered pair (r, θ), where

- $r \geq 0$ is the **radial distance** which is the distance from the origin to the point, and
- θ is the **polar angle**, measured counterclockwise from the positive x-axis.

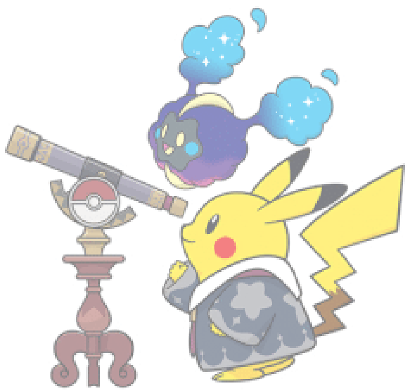
Given polar coordinates (r, θ), the corresponding Cartesian coordinates satisfy

$$x = r\cos\theta, \qquad y = r\sin\theta$$

Given Cartesian coordinates (x, y), the corresponding polar coordinates satisfy

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

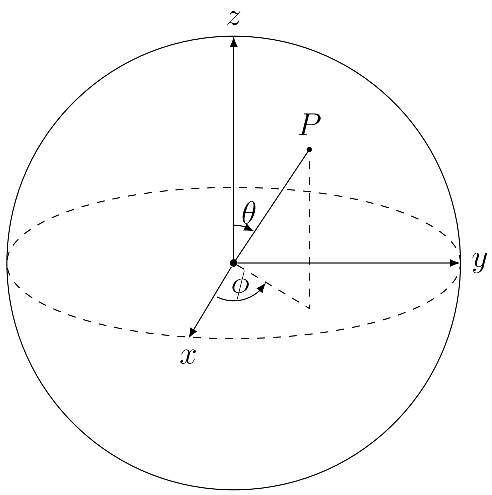
with the correct quadrant of θ chosen according to the signs of x and y.



2 Spherical Trigonometry

2.1 Spherical Coordinates

This section is adapted from notes by 张植竣, a former member of the IOAA China team and currently a PhD researcher at the National Astronomical Observatories of China. The author kindly provides academic advice and has granted permission for the inclusion of this material. The original notes are available at: https://github.com/Firestar-Reimu-Pro/My-LaTeX-Files/tree/main.



A point in spherical coordinates is specified by

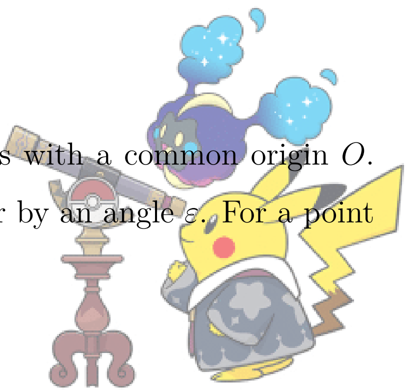
$$(\rho, \theta, \phi)$$

where

- $\rho \geq 0$ is the radial distance from the origin,
- $\theta \in [0, \pi]$ is the polar angle measured from the positive z-axis,
- $\phi \in [0, 2\pi)$ is the azimuthal angle measured in the xy-plane from the positive x-axis.

2.2 Law of Sine and Law of Cosine

Let $O-xyz$ and $O-x'y'z'$ be two right-handed Cartesian coordinate systems with a common origin O. The y-axes coincide, while the x- and z-axes are rotated relative to each other by an angle ε. For a point



P on the sphere, its spherical coordinates in the two systems are

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta, \tag{5}$$

$$x' = r\sin\theta'\cos\phi', \qquad y' = r\sin\theta'\sin\phi', \qquad z' = r\cos\theta' \tag{6}$$

Hence,

$$\begin{cases} r\sin\theta\cos\phi = r\sin\theta'\cos\phi' \\ r\sin\theta\sin\phi = r\sin\theta'\sin\phi' \\ r\cos\theta = r\cos\theta' \end{cases} \tag{7}$$

By geometry, the coordinates (x, z) and (x', z') satisfy

$$\begin{cases} x' = x\cos\varepsilon + z\sin\varepsilon \\ z' = z\cos\varepsilon - x\sin\varepsilon \end{cases} \tag{8}$$

Substituting the spherical-coordinate expressions into Eqs. (7) and (8), we obtain

$$\begin{cases} \sin\theta'\cos\phi' = \sin\theta\cos\phi\cos\varepsilon + \cos\theta\sin\varepsilon \\ \sin\theta'\sin\phi' = \sin\theta\sin\phi \\ \cos\theta' = \cos\theta\cos\varepsilon - \sin\theta\cos\phi\sin\varepsilon \end{cases} \tag{9}$$

Let the intersection points of the positive z-axis and z'-axis with the sphere, together with point P, form a spherical triangle. Define

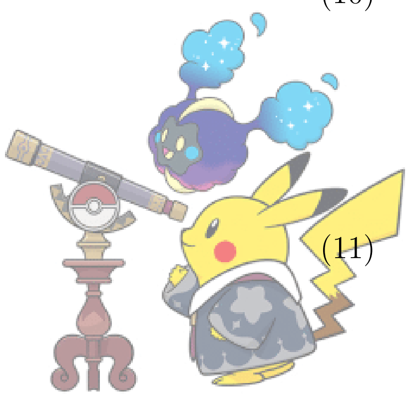
$$a = \theta', \quad b = \theta, \quad c = \varepsilon, \quad A = \pi - \phi, \quad B = \phi'$$

Then Eq. (9) becomes

$$\begin{cases} \sin a\cos B = \cos b\sin c - \cos A\sin b\cos c \\ \sin A\sin b = \sin a\sin B \\ \cos a = \cos b\cos c + \cos A\sin b\sin c \end{cases} \tag{10}$$

From Eq. (10), the spherical law of sines follows:

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \tag{11}$$



The cosine rule for sides is

$$\cos a = \cos b \cos c + \cos A \sin b \sin c \tag{12}$$

An alternative form, the cosine rule for angles, is

$$\cos A = -\cos B \cos C + \cos a \sin B \sin C \tag{13}$$

If $A = \frac{\pi}{2}$, then the cosine rule simplifies to

$$\cos a = \cos b \cos c \tag{14}$$

When the sides a, b, and c are small, the spherical triangle approaches a plane triangle. Using the second order approximation

$$\sin a \approx a, \qquad \cos a \approx 1 - \frac{a^2}{2}$$

Eq. (10) reduces to

$$\begin{cases} \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} \\ c^2 = a^2 + b^2 - 2ab\cos A \end{cases} \tag{15}$$

They are precisely the law of sines and the law of cosines in plane geometry.

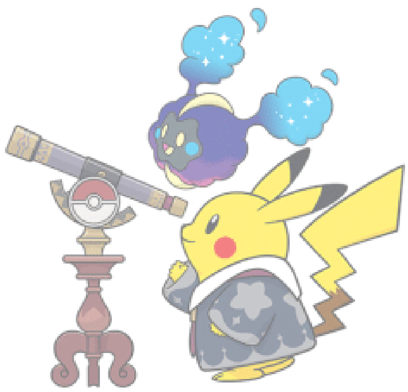
3 Astronomical Coordinate Systems

3.1 Horizontal Coordinate System

The **Horizontal Coordinate System**, also known as the **Altitude–Azimuth (Alt–Az) Coordinate System**, is a location-based system used in observational astronomy to specify the position of a celestial object in the sky **as seen by a particular observer at a particular time**. In the coordinate system, there are

- **Horizon plane**: the plane perpendicular to the local vertical direction.
- **Zenith**: the point directly above the observer.
- **Cardinal directions**: North (N), East (E), South (S), West (W).

A celestial object's position is specified by two angles:



1. **Altitude** (h or Alt): **Altitude** is the angular distance of an object above or below the horizon.

2. **Azimuth** (A or Az): **Azimuth** is the angular direction of an object measured positively along the horizon from the North.

3.2 Equatorial Coordinate System

The **Equatorial coordinate system** is used to specify the positions of celestial objects on the celestial sphere in a way that is largely **independent of the observer's location** and only slowly changes with time.

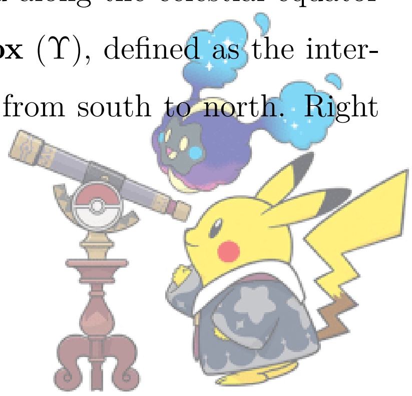
To define the equatorial coordinate system, we introduce the **celestial sphere**, an imaginary sphere of very large radius centered on the Earth. Also, there are some other features:

- **Celestial poles**: They are the extensions of Earth's rotation axis intersecting the celestial sphere.
- **Celestial equator**: It is the projection of Earth's equator onto the celestial sphere.
- **Celestial meridian**: It is the great circle passing through the celestial poles and the zenith of the observer.

The position of a celestial object is specified by two angles:

- **Declination** (δ): It is the angular distance of an object north or south of the celestial equator. Note that
  - $\delta = 0°$: object lies on the celestial equator
  - $\delta > 0°$: object is north of the celestial equator
  - $\delta < 0°$: object is south of the celestial equator
  - $\delta = +90°$: north celestial pole
  - $\delta = -90°$: south celestial pole
- **Right Ascension** (α): It is the angular distance of an object eastward along the celestial equator from a fixed reference point. The reference point is the **vernal equinox** (Υ), defined as the intersection of the celestial equator with the ecliptic where the Sun crosses from south to north. Right ascension is usually measured in **time units**:

$$0 \leq \alpha < 24^{\mathrm{h}}$$



The Earth rotates 360° in approximately 24 hours. Measuring right ascension in hours makes it directly related to Earth's rotation and the apparent daily motion of the sky.
For example, $360^\circ = 24$ h.

Equatorial coordinates are fixed on the celestial sphere, but an observer sees objects in horizontal coordinates (altitude and azimuth). The transformation between equatorial coordinates (α, δ) and horizontal coordinates (A, h) depends on:

- Observer's latitude ϕ
- Local sidereal time: When a celestial object crosses the observer's meridian, its right ascension equals the **local sidereal time**.
- Hour angle H: It is defined as

$$H = \text{LST} - \alpha$$

A key relation is:

$$\sin h = \sin\phi \sin\delta + \cos\phi \cos\delta \cos H$$

3.3 Ecliptic Coordinate System

3.3.1 Ecliptic Plane

The **ecliptic plane** is the plane of the Earth's orbit around the Sun. Its projection onto the celestial sphere defines a great circle called the **ecliptic**.

3.3.2 Obliquity of the Ecliptic

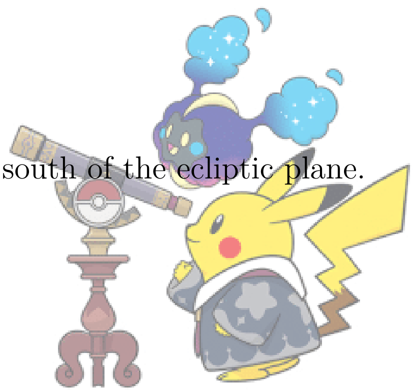
The Earth's rotation axis is tilted relative to the ecliptic plane by an angle called the **obliquity of the ecliptic**:

$$\varepsilon \approx 23.44^\circ$$

3.3.3 Coordinate Definitions

A celestial object's position in the ecliptic system is specified by:

- **Ecliptic latitude** (β): It is the angular distance of an object north or south of the ecliptic plane.



- **Ecliptic longitude** (λ): It is measured eastward along the ecliptic from the vernal equinox. Historically, ecliptic longitude was divided into twelve 30° segments corresponding to the zodiac constellations.

Given equatorial coordinates (α, δ), the corresponding ecliptic coordinates (λ, β) are obtained by a rotation by the obliquity angle ε.

$$\sin\beta = \sin\delta\cos\varepsilon - \cos\delta\sin\varepsilon\sin\alpha$$

$$\tan\lambda = \frac{\sin\alpha\cos\varepsilon + \tan\delta\sin\varepsilon}{\cos\alpha}$$

3.4 Galactic Coordinate System

3.4.1 Galactic Plane

The **galactic plane** is the plane defined by the mid-plane of the Milky Way disk. The Sun lies very close to this plane, making it a convenient reference.

3.4.2 Galactic Center

The **galactic center** is the point in the sky towards which the center of the Milky Way lies, located in the constellation Sagittarius. This point is used as the zero point for galactic longitude.

3.4.3 North Galactic Pole

The **north galactic pole** (NGP) is the point on the celestial sphere perpendicular to the galactic plane. Its approximate position in equatorial coordinates (J2000.0) is:

$$\alpha_{\mathrm{NGP}} = 12^{\mathrm{h}}51^{\mathrm{m}}, \quad \delta_{\mathrm{NGP}} = +27^\circ 07'$$

3.4.4 Coordinate Definitions

A celestial object's position in the galactic system is specified by:

- **Galactic longitude** (l): It is measured along the galactic plane from the direction of the galactic center.
- **Galactic latitude** (b): It is the angular distance from the galactic plane.

Let $(\alpha_{\text{NGP}}, \delta_{\text{NGP}})$ be the north galactic pole in equatorial coordinates, and α_0 the RA of the galactic center. Then

$$\sin b = \sin\delta \sin\delta_{\text{NGP}} + \cos\delta \cos\delta_{\text{NGP}} \cos(\alpha - \alpha_{\text{NGP}})$$

$$\tan(l - l_0) = \frac{\cos\delta \sin(\alpha - \alpha_{\text{NGP}})}{\sin\delta \cos\delta_{\text{NGP}} - \cos\delta \sin\delta_{\text{NGP}} \cos(\alpha - \alpha_{\text{NGP}})}$$

where $l_0 = 0^\circ$ is the galactic longitude of the galactic center.

3.5 Circumpolar Stars

Circumpolar stars are stars that never set below the horizon from a given location. Their position is close enough to one of the celestial poles that they appear to trace circular paths around the pole throughout the night. These stars are visible year-round in regions near the pole, such as in the northern hemisphere where the North Star (α Ursae Minoris) is circumpolar. Mathematically, the declination δ of a star must satisfy:

$$\delta \geq 90^\circ - \phi$$

where ϕ is the latitude of the observer.

4 Astronomical Time

4.1 Solar Time

4.1.1 Definition

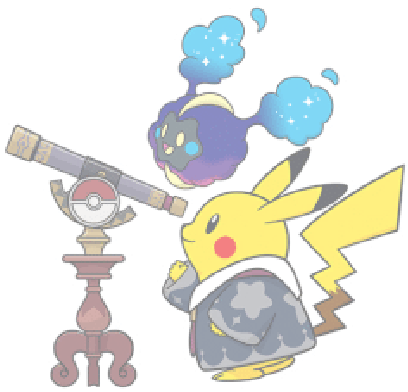
Solar time is based on the apparent motion of the Sun across the sky. It measures time by the Earth's rotation relative to the Sun.

4.1.2 Apparent Solar Time

Apparent solar time is determined directly by the observed position of the real Sun. It can be measured using a sundial.However, apparent solar days are not all of equal length due to:

1. The ellipticity of the Earth's orbit (Kepler's second law).
2. The obliquity of the Earth's axis relative to the ecliptic.

As a result, the apparent solar day varies slightly over the year.



4.1.3 Mean Solar Time

To obtain a uniform time scale, astronomers define the **mean Sun**, a fictitious point that moves uniformly along the celestial equator, completing one revolution per year. **Mean solar time** is the time measured by the hour angle of the mean Sun.

- A **mean solar day** is defined to be exactly 24 hours.
- Civil time (e.g. UTC, ignoring leap seconds) is based on mean solar time.

4.1.4 Equation of Time

The difference between apparent solar time and mean solar time is called the **equation of time**:

$$\text{Equation of Time} = \text{Apparent Solar Time} - \text{Mean Solar Time}$$

Its value varies throughout the year and can be as large as about ± 16 minutes.

4.2 Sidereal Time

4.2.1 Definition

Sidereal time is based on the Earth's rotation relative to the distant stars rather than the Sun. **Local sidereal time (LST)** is defined as the hour angle of the vernal equinox.

4.2.2 Sidereal Day

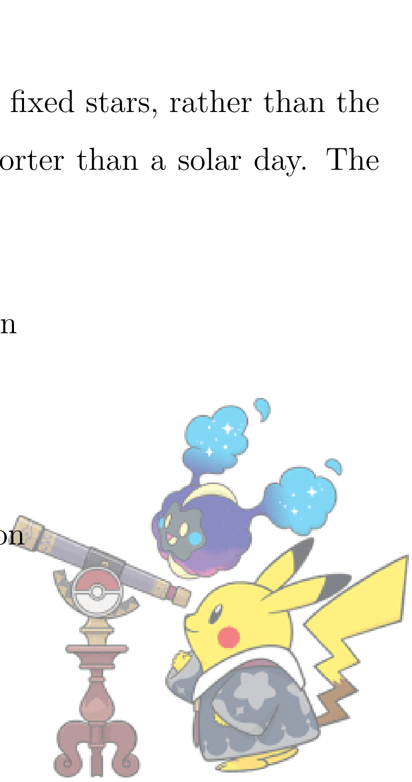
A sidereal day is the time it takes for the Earth to rotate once relative to the fixed stars, rather than the Sun. This period is about 23 hours, 56 minutes, and 4.1 seconds, slightly shorter than a solar day. The relationship between solar time and sidereal time is given by

$$\text{Sidereal Time} = \text{Solar Time} - \text{Longitude Correction}$$

To calculate sidereal time at a given location, the following formula is used:

$$\text{Sidereal Time (ST)} = \text{GMST} + \text{Longitude Correction}$$

where



- **GMST** is the Greenwich Mean Sidereal Time, which is the sidereal time at the Prime Meridian.
- **Longitude Correction** is based on the observer’s longitude.

The GMST is computed as follows:

$$\text{GMST} = 280.46061837^\circ + 360.98564736629^\circ \times (\text{JD} - 2451545.0)$$

where JD is the Julian Date. The **Julian Date (JD)** is the number of days (and fractions of a day) that have elapsed since

12:00 noon (UT) on 1 January 4713 BCE (Julian calendar)

4.3 Heliocentric Julian Date (HJD)

The **Heliocentric Julian Date** is the Julian Date (JD) corrected for the light travel time between the Earth and the Sun. This correction accounts for the position of the Earth in its orbit.

$$\text{HJD} = \text{JD} + \frac{\mathbf{r} \cdot \hat{s}}{c}$$

where

- $\mathbf{r}$ is the vector from the Sun to the Earth,
- $\hat{s}$ is the unit vector pointing toward the observed object,
- c is the speed of light.

4.4 Local Mean Time (LMT)

Local Mean Time (LMT) at a given location is defined as the mean solar time at that longitude.

4.5 Universal Time (UT)

Universal Time (UT) is a global time standard defined as the Local Mean Time at the Prime Meridian (0° longitude), which passes through Greenwich, London. Hence,

$$\text{UT} = \text{LMT at } \lambda = 0^\circ$$

Universal Time is therefore often referred to as **Greenwich Mean Time (GMT)** in non-technical contexts.

4.6 Time Zones

To simplify civil timekeeping, the Earth is divided into **time zones**, each using a single standard time rather than local mean time at every longitude. In an ideal model:

- The Earth is divided into 24 time zones.
- Each zone spans 15° of longitude.
- The central meridian of each zone differs from the next by 15°.

The standard time of a zone equals the LMT of its central meridian. If a time zone has central longitude λ_0, then its standard time satisfies

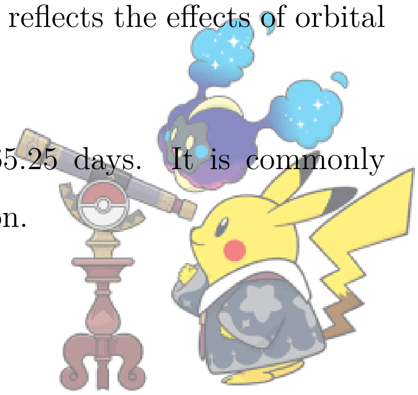
$$\text{Zone Time} = \text{UT} + \frac{\lambda_0}{15}$$

This offset is usually written as UTC $\pm n$, where n is an integer number of hours.

4.7 Different Definitions of a Year

There are several distinct astronomical definitions of a year, each corresponding to a different reference frame or physical phenomenon:

- **Sidereal Year:** The time required for the Earth to complete one full revolution around the Sun with respect to the fixed stars. Its length is approximately 365.25636 days.
- **Tropical Year:** The interval between two successive passages of the Sun through the vernal equinox. With a duration of approximately 365.24219 days, it is the basis of civil calendars, as it ensures long-term alignment with the seasons.
- **Anomalistic Year:** The time elapsed between two successive passages of the Earth through perihelion. This year has a mean length of approximately 365.25964 days and reflects the effects of orbital precession.
- **Julian Year:** A conventional unit of time defined to be exactly 365.25 days. It is commonly employed in astronomical calculations for simplicity and standardization.



5 Basic Units in Astronomy

Definition. 5.1: Astronomical Unit

The **Astronomical Unit (AU)** is the average distance between the Earth and the Sun.

Definition. 5.2: Light Year

A **light-year** is the distance that light travels in a vacuum in one year.

Definition. 5.3: Parsec

A **parsec** is the distance at which one astronomical unit subtends an angle of one arcsecond.

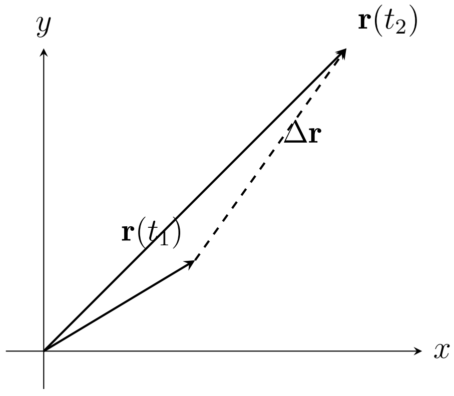
$$1\,\text{pc} = \frac{1\,\text{AU}}{\tan(1'')} \approx 3.086 \times 10^{16}\,\text{m} \approx 3.262\,\text{ly} \approx 206,265\,\text{AU}$$

6 Fundamental Mechanics

6.1 Kinematics

6.1.1 Linear Motion

We can use the **position vector** $\mathbf{r}(t)$ to specify the position of a particle in a plane at time t.



The figure above shows the position of a particle at two different times, t_1 and t_2. The **displacement vector** is given by

$$\Delta\mathbf{r} = \mathbf{r}(t_2) - \mathbf{r}(t_1)$$

The **average velocity** during the time interval $\Delta t = t_2 - t_1$ is:

$$\mathbf{v}_{\text{av}} = \frac{\Delta \mathbf{r}}{\Delta t}$$

The **instantaneous velocity** at time t_0 is obtained by taking the limit as $\Delta t \to 0$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

The **instantaneous speed** is the magnitude of the velocity vector

$$v = |\mathbf{v}|$$

The **average acceleration** is the rate of change of velocity

$$\mathbf{a}_{\text{av}} = \frac{\Delta \mathbf{v}}{\Delta t}$$

and the **instantaneous acceleration** is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$$

Suppose a particle moves with a **constant acceleration a**. At time t_0, let its velocity and position be $\mathbf{v}_0$ and $\mathbf{x}_0$, respectively. Then, the motion can be described by:

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{v}_0(t - t_0) + \tfrac{1}{2}\mathbf{a}(t - t_0)^2$$

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{a}(t - t_0)$$

$$|\mathbf{v}|^2 - |\mathbf{v}_0|^2 = 2\,\mathbf{a} \cdot (\mathbf{x} - \mathbf{x}_0)$$

The time derivative of a vector provides information about the rate of change of the vector's magnitude and direction.

6.1.2 Relative Motion

Introduction Let us consider two objects, A and B, moving in a straight line or in a plane. The velocity of object A relative to object B is given by

$$\mathbf{v}_{A/B} = \mathbf{v}_A - \mathbf{v}_B$$

where

- $\mathbf{v}_A$ is the velocity of object A relative to a fixed reference frame.
- $\mathbf{v}_B$ is the velocity of object B relative to the same reference frame.
- $\mathbf{v}_{A/B}$ is the velocity of object A relative to object B.

When two bodies are rotating relative to each other, their angular velocities need to be combined to determine the relative angular velocity. If object A has angular velocity $\boldsymbol{\omega}_A$ and object B has angular velocity $\boldsymbol{\omega}_B$, the relative angular velocity of A with respect to B, denoted as $\boldsymbol{\omega}_{A/B}$, is given by:

$$\boldsymbol{\omega}_{A/B} = \boldsymbol{\omega}_A - \boldsymbol{\omega}_B$$

It is assumed that both objects are rotating about the same axis.

Retrograde Motion Retrograde motion is the apparent backward motion of a planet relative to the background stars, caused by the relative orbital motion of Earth and the planet. Earth moves faster in its inner orbit. Earth moves faster in its inner orbit.

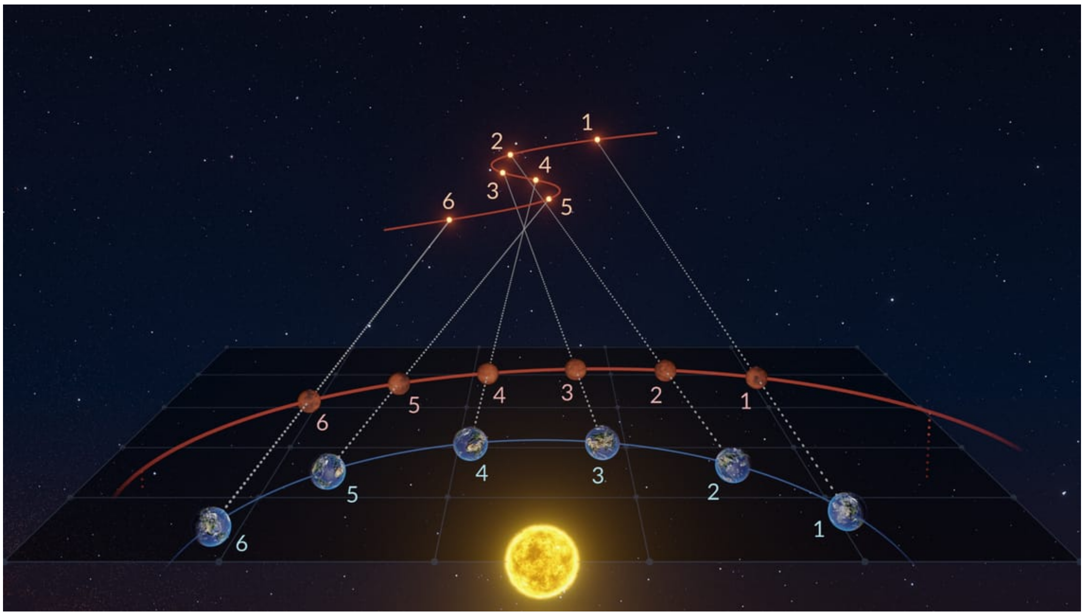
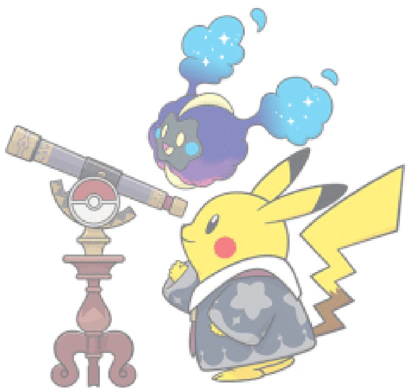


Figure 4: Source: https://starwalk.space/en/news/what-is-retrograde-motion

Example. Given:

$$T_E = 365.25 \text{ days}$$

$$T_{Mars} = 686.98 \text{ days}$$



Angular speeds:

$$\omega_E = \frac{360^\circ}{365.25} = 0.9856^\circ/\text{day}$$
$$\omega_{Mars} = \frac{360^\circ}{686.98} = 0.524^\circ/\text{day}$$

Relative angular speed:

$$\omega_{rel} = \omega_E - \omega_{Mars} = 0.4616^\circ/\text{day}$$

This relative angular motion causes Mars to appear to move backward against the stars during opposition. The duration between successive retrograde motions of a planet is equal to its **synodic period**.

Synodic Period and Sidereal Period The **synodic period** (T_{syn}) is the time interval between two successive identical configurations of a planet as observed from Earth (e.g., opposition to opposition or conjunction to conjunction).

The **sidereal period** (T_s) of a celestial body is the time taken to complete one full revolution around the Sun with respect to the fixed background stars.

Example. Let

- T_E = sidereal period of Earth
- T_P = sidereal period of the planet

For **inferior planets** (Mercury, Venus):

$$\frac{1}{T_{syn}} = \frac{1}{T_P} - \frac{1}{T_E}$$

For **superior planets** (Mars, Jupiter, Saturn):

$$\frac{1}{T_{syn}} = \frac{1}{T_E} - \frac{1}{T_P}$$

Slingshot Effect of Gravity The **slingshot effect**, also known as **gravitational assist**, occurs when a spacecraft uses the gravity of a planet or moon to change its trajectory and increase its velocity. This is achieved by flying close to a planet or moon, which causes the spacecraft to accelerate as it is pulled by the planet's gravitational field. The effect is used to save fuel and to direct spacecraft to their destinations.

Mathematically, this can be described by the following equation for velocity change during the flyby

$$\Delta v = v_{\text{spacecraft}} + v_{\text{planet}}$$

where

1. $v_{\text{spacecraft}}$ is the velocity of the spacecraft, and
2. v_{planet} is the velocity of the planet as seen from the spacecraft's reference frame.

6.2 Newton's Law

First Law (Law of Inertia) An object remains at rest, or moves in a straight line at constant velocity, unless acted upon by an external force.

$$\sum \mathbf{F} = 0 \implies \mathbf{a} = 0$$

Second Law The net force on a particle is equal to the time rate of change of its momentum:

$$\sum \mathbf{F} = \frac{d\mathbf{p}}{dt}$$

For constant mass,

$$\sum \mathbf{F} = m\mathbf{a}$$

This is the most commonly used form of Newton’ s Second Law.

Third Law For every action, there is an equal and opposite reaction:

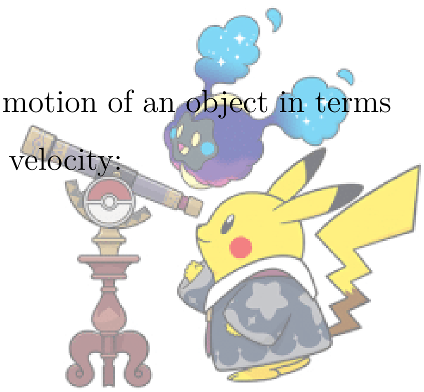
$$\mathbf{F}_{AB} = -\mathbf{F}_{BA}$$

6.3 Linear Momentum

6.3.1 Introduction

Linear momentum is a fundamental concept in mechanics that describes the motion of an object in terms of its mass and velocity. It is defined as the product of an object's mass and velocity:

$$\mathbf{p} = m\mathbf{v}$$



where

- $\mathbf{p}$ is the linear momentum of the object,
- m is the mass of the object, and
- $\mathbf{v}$ is the velocity of the object.

6.3.2 Conservation of Linear Momentum

In an isolated system (no external forces), the total linear momentum of the system remains constant. This is known as the conservation of linear momentum. Mathematically, for a system of particles,

$$\sum_{i=1}^{n} \mathbf{p}_i = \text{constant}$$

where $\mathbf{p}_i$ is the linear momentum of the i-th particle.

6.3.3 Impulse

The impulse of a force $\mathbf{F}$ acting on an object is defined as the change in the object's momentum. Mathematically,

$$\mathbf{J} = \Delta\mathbf{p} = \mathbf{F}\Delta t$$

where $\mathbf{J}$ is the impulse, $\Delta\mathbf{p}$ is the change in momentum, $\mathbf{F}$ is the average force, and Δt is the time interval during which the force acts.

6.3.4 Rocket Equation

Let the rocket expel a small mass of fuel dm in a short time interval dt. By conservation of momentum,

$$\text{Initial momentum} = \text{Final momentum}$$

Initially, the momentum of the rocket is

$$p_{\text{initial}} = m(t)v(t)$$

After expelling dm of fuel, the rocket mass becomes $m(t) - dm$ and its velocity becomes $v(t) + dv$. The expelled fuel has momentum $dm \cdot (v - u)$, because it moves at velocity $v - u$ in the inertial frame. Hence

the final momentum is

$$p_{\text{final}} = \big(m(t) - dm\big)\big(v + dv\big) + dm\big(v - u\big)$$

Using differential approximation (ignoring higher-order terms $dm\,dv$) and equating momentum:

$$m\,dv = u\,dm$$

Integrating from the initial mass m_0 to the final mass m_f and initial velocity v_0 to final velocity v_f:

$$\int_{v_0}^{v_f} dv = u \int_{m_0}^{m_f} \frac{dm}{m}$$

$$\boxed{v_f - v_0 = u \ln \frac{m_0}{m_f}}$$

If external forces (like gravity g or drag F_{drag}) act on the rocket, the generalized form is

$$m\frac{dv}{dt} = u\frac{dm}{dt} + F_{\text{ext}}(t)$$

where $F_{\text{ext}}(t)$ represents any additional force.

6.4 Force, Torque and Equilibrium of Body

The force is any interaction that causes a change in an object's motion. It is a vector quantity, having both magnitude and direction. In vector notation:

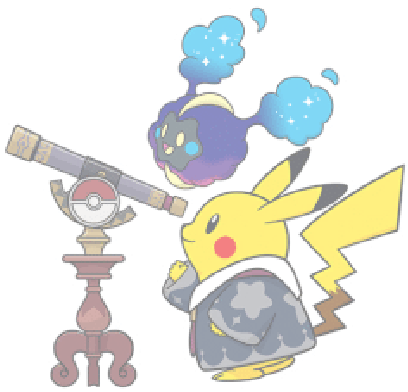
$$\mathbf{F} = (F_x, F_y, F_z)$$

SI Unit: Newton (N) which is defined as

$$1\,\text{N} = 1\,\text{kg m s}^{-1}$$

Example of Forces

- Gravitational force: $\mathbf{F}_g = m\mathbf{g}$
- Normal force: the perpendicular reaction from a surface
- Friction: $\mathbf{f} = \mu N$ (opposes motion)



- Tension: force transmitted through a rope

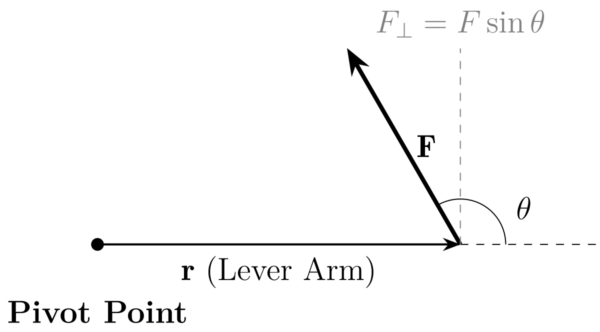
The torque $\boldsymbol{\tau}$ produced by a force $\mathbf{F}$ acting at a distance $\mathbf{r}$ from the pivot is the cross product:

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

The magnitude is given by

$$\tau = rF\sin(\theta)$$

SI Unit: Newton-meter (Nm)



A body is said to be in equilibrium when the net force acting on it is zero, and the net torque acting on it is also zero. There are two types of equilibrium:

- **Translational equilibrium** occurs when the sum of all forces acting on a body is zero:

$$\sum \mathbf{F} = 0$$

- **Rotational equilibrium** occurs when the sum of all torques acting on a body about any point is zero:

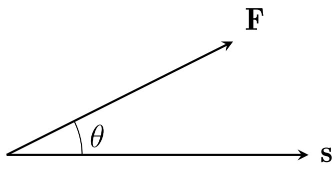
$$\sum \boldsymbol{\tau} = 0$$

6.5 Energy

When a constant force $\mathbf{F}$ acts on an object and causes a displacement $\mathbf{s}$, the work done by the force is defined as the dot product

$$W = \mathbf{F} \cdot \mathbf{s} = Fs\cos\theta$$

where θ is the angle between the force and the displacement.



If the force is not constant, then

$$W = \int_C \mathbf{F} \cdot d\mathbf{s}$$

where C is the path of the object. Note that

- Work is a **scalar quantity**.
- $W > 0$ if force and displacement are in the same direction.
- $W < 0$ if force opposes motion (e.g. friction).
- $W = 0$ if $\mathbf{F} \perp \mathbf{s}$ (no work done).

For an object of mass m moving vertically through a height h:

$$W_g = -mgh$$

if the object is lifted upward, and

$$W_g = +mgh$$

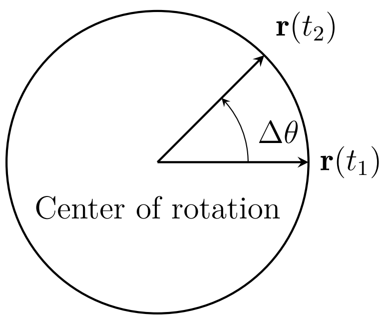
if it moves downward. The **kinetic energy** of a body of mass m moving with speed v is

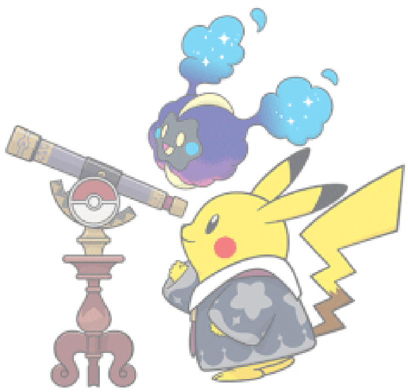
$$K = \frac{1}{2}mv^2$$

6.6 Circular Motion

6.6.1 Kinematics

Consider an object (for example, a dog on a rope) rotating about a fixed point with a radius r.





- **Angular displacement:** $\Delta\theta$ (in radians)
- **Angular velocity:** $\omega = \dfrac{d\theta}{dt}$
- **Angular acceleration:** $\alpha = \dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2}$

The rotational kinematic equations (analogous to linear motion) are

$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2$$

$$\omega = \omega_0 + \alpha t$$

$$\omega^2 - \omega_0^2 = 2\alpha(\theta - \theta_0)$$

For an object moving in a circle of radius r,

$$v = r\omega$$

is the **tangential (linear) speed**.

The **tangential acceleration** is

$$a_{\text{tan}} = r\alpha$$

and the **radial (centripetal) acceleration** is

$$a_{\text{rad}} = \frac{v^2}{r} = r\omega^2$$

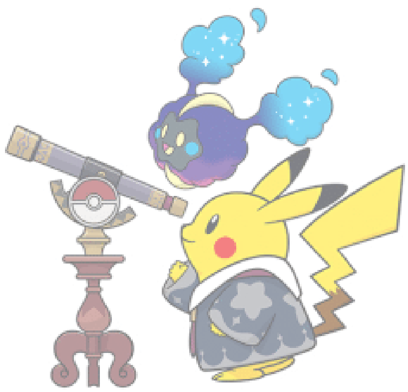
6.6.2 Centripetal Force

Centripetal force is the force that acts on an object moving in a circular path. It is directed towards the center of the circle or axis of rotation. The magnitude of the centripetal force is given by

$$F_c = \frac{mv^2}{r}$$

where

- m is the mass of the object,
- v is the velocity of the object,
- r is the radius of the circular path.



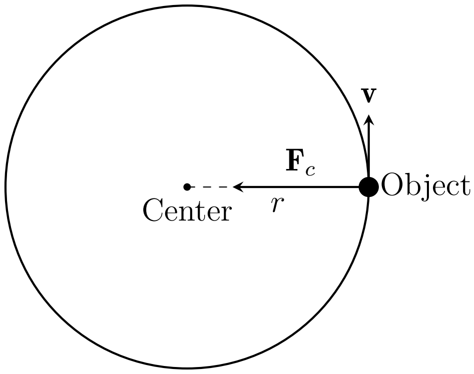


Figure 5: Centripetal force always points toward the center of the circle.

6.6.3 Centrifugal Force

Centrifugal force is a fictitious or pseudo force that appears to act on an object when viewed from a rotating reference frame. It appears to push the object away from the center of rotation. It is equal in magnitude and opposite in direction to the centripetal force in the rotating frame of reference. The magnitude of the centrifugal force is

$$F_{\text{cf}} = \frac{mv^2}{r}$$

6.6.4 Coriolis Force

Coriolis force is an apparent force that acts on a mass moving in a rotating system, such as the Earth. It arises due to the rotation of the reference frame. The Coriolis force is given by

$$F_{\text{Coriolis}} = 2m\mathbf{v} \times \boldsymbol{\omega}$$

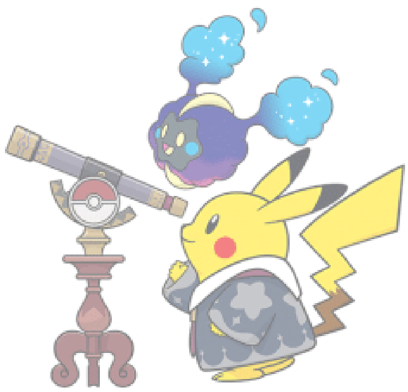
where

- m is the mass of the object,
- $\mathbf{v}$ is the velocity of the object relative to the rotating frame,
- $\boldsymbol{\omega}$ is the angular velocity vector of the rotating reference frame.

6.6.5 Angular Momentum

For a particle at position $\mathbf{r}$ (relative to the origin) with linear momentum $\mathbf{p}$:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$



For a rigid body rotating around a fixed axis with moment of inertia I and angular velocity ω:

$$\mathbf{L} = I\omega$$

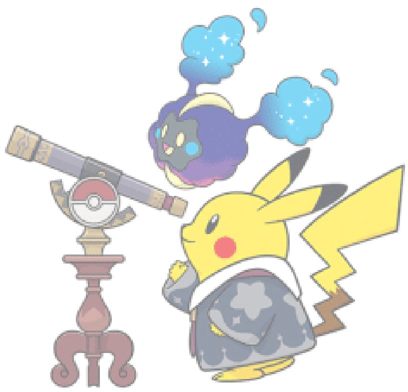
Just as force changes linear momentum, torque ($\boldsymbol{\tau}$) changes angular momentum:

$$\boldsymbol{\tau}_{net} = \frac{d\mathbf{L}}{dt}$$

Theorem. 6.1: Conservation of Angular Momentum

If the net external torque on a system is zero ($\boldsymbol{\tau}_{ext} = 0$), the total angular momentum is conserved:

$$L_i = L_f$$



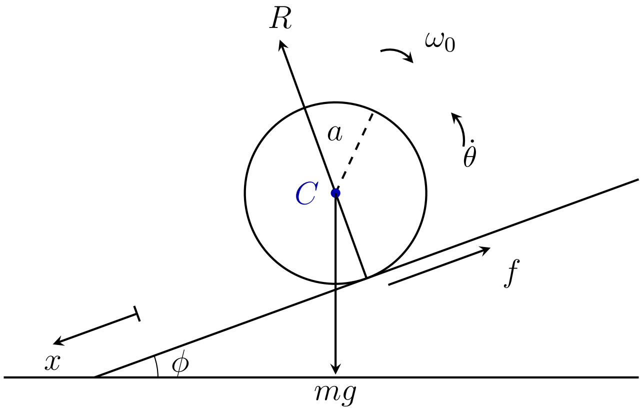
6.7 Case Study

A circular disk rotating about its axis with angular speed ω_0 is placed gently with its axis horizontal on a rough and inclined plane such that the friction acts up the plane. Given that the coefficient of friction is μ and the inclination angle of the plane to the horizontal is ϕ.

(a) Show that the disk will move upwards if $\mu > \tan\phi$.

(b) Find the time that elapses before rolling takes place.

Solution. (a)



Equations of Motion: Take downward as positive and take counterclockwise as positive. Using Newton's Second Law for translation along the plane and rotation about the center of mass:

$$m\ddot{x} = mg\sin\phi - f \tag{16}$$

$$R - mg\cos\phi = 0 \implies R = mg\cos\phi \tag{17}$$

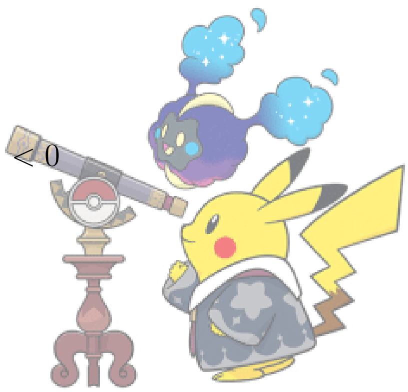
$$I\ddot{\theta} = \tau \tag{18}$$

Since the disk is slipping, kinetic friction applies:

$$f = \mu R = \mu mg\cos\phi \tag{19}$$

Substituting (19) into (16):

$$m\ddot{x} = mg\sin\phi - \mu mg\cos\phi \implies \ddot{x} = g(\sin\phi - \mu\cos\phi) < 0$$



$$g(\sin\phi - \mu\cos\phi) < 0$$

$$\sin\phi < \mu\cos\phi$$

$$\tan\phi < \mu$$

Hence, the disk will move upwards if $\boxed{\mu > \tan\phi}$.

(b) For pure rolling,

$$v_{\text{contact}} = v + a\omega = 0$$

where $v = \dot{x}$ is the linear velocity and $\omega = \dot{\theta}$ is the angular velocity. Note that

$$v(t) = v_0 + \ddot{x}t = 0 + g(\sin\phi - \mu\cos\phi)t$$

Also,

$$\tau = -f \cdot a = -(\mu mg\cos\phi)a$$

Using $I = \frac{1}{2}ma^2$ for a disk,

$$\frac{1}{2}ma^2\ddot{\theta} = -\mu mga\cos\phi \implies \ddot{\theta} = -\frac{2\mu g\cos\phi}{a}$$

Note that

$$\omega(t) = \omega_0 + \ddot{\theta}t = \omega_0 - \frac{2\mu g\cos\phi}{a}t$$

Substitute (6.7) and (6.7) into the rolling condition (6.7):

$$v(t) + a\omega(t) = 0$$

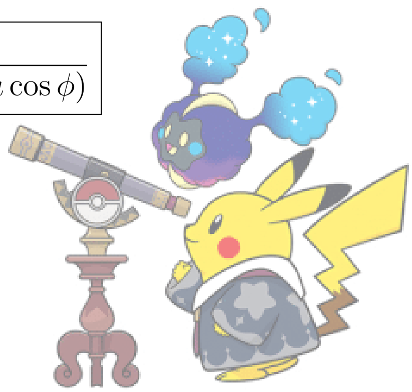
$$[g(\sin\phi - \mu\cos\phi)t] + a\left[\omega_0 - \frac{2\mu g\cos\phi}{a}t\right] = 0$$

$$gt(\sin\phi - \mu\cos\phi) + a\omega_0 - 2\mu gt\cos\phi = 0$$

$$gt(\sin\phi - \mu\cos\phi - 2\mu\cos\phi) = -a\omega_0$$

$$gt(\sin\phi - 3\mu\cos\phi) = -a\omega_0$$

$$t = \boxed{\frac{-a\omega_0}{g(\sin\phi - 3\mu\cos\phi)}}$$



7 Orbital Mechanics

7.1 Newton's Law of Universal Gravitation

7.1.1 Introduction

Theorem. 7.1: Newton's Law of Universal Gravitation

The Universal Law of Gravity states that every point mass attracts every other point mass with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centers. This is mathematically expressed as

$$F = G\frac{m_1 m_2}{r^2}$$

where

- F is the magnitude of the gravitational force between the two masses,
- G is the gravitational constant,
- m_1 and m_2 are the two masses,
- r is the distance between the centers of the two masses.

Theorem. 7.2: Newton's Law of Universal Gravitation (Vector Form)

The gravitational force between two point masses m_1 and m_2 can be written in vector form as

$$\mathbf{F} = -G\frac{m_1 m_2}{r^2}\hat{r}$$

where

- $\mathbf{F}$ is the gravitational force vector on m_1 due to m_2,
- G is the gravitational constant,
- m_1 and m_2 are the masses of the two objects,
- r is the distance between the two masses,
- $\hat{r}$ is the unit vector pointing from m_1 to m_2, i.e., the direction of the force.

The negative sign indicates that the force is attractive, meaning the force is directed towards the other mass.

Theorem. 7.3: Gauss's Law of Gravity

$$\nabla \cdot \mathbf{g} = -4\pi G\rho$$

where

- $\nabla \cdot \mathbf{g}$ is the divergence of the gravitational field $\mathbf{g}$, and
- ρ is the mass density of the object.

This law can be integrated over a closed surface to give the integral form of Gauss's Law for gravity:

$$\oint_S \mathbf{g} \cdot d\mathbf{A} = -4\pi G M_{\text{enc}}$$

where M_{enc} is the mass enclosed by the surface.

7.1.2 Barycentre

The **barycentre** or **centre of mass** is the weighted average position of all the mass in a system. It is the point where, if all the mass were concentrated, the system would balance. For a system of particles with masses $m_1, m_2, \ldots, m_n$ located at positions $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$, the position of the barycentre $\mathbf{R}$ is given by

$$\mathbf{R} = \frac{1}{M}\sum_{i=1}^{n} m_i \mathbf{r}_i$$

where $M = \sum_{i=1}^{n} m_i$ is the total mass of the system.
For a continuous mass distribution, the barycentre is calculated by integrating over the mass distribution. If the mass density is $\rho(\mathbf{r})$, the barycentre position is given by

$$\mathbf{R} = \frac{1}{M}\int_V \mathbf{r}\,\rho(\mathbf{r})\,dV$$

where $M = \int_V \rho(\mathbf{r})\,dV$ is the total mass of the system, and the integral is taken over the volume V of the mass distribution.

7.1.3 2-body Problem

Consider two point masses, m_1 at position vector $\mathbf{r}_1$ and m_2 at position vector $\mathbf{r}_2$, interacting via a central force:

$$m_1\ddot{\mathbf{r}}_1 = \mathbf{F}_{21}, \quad m_2\ddot{\mathbf{r}}_2 = \mathbf{F}_{12} = -\mathbf{F}_{21}$$

The center of mass (barycenter) is located at

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

and since the internal forces cancel by Newton's third law, the acceleration of the center of mass is

$$\ddot{\mathbf{R}} = 0$$

Hence, the center of mass moves in a straight line with constant velocity. Define the relative position vector

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

Differentiating twice with respect to time gives:

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \frac{\mathbf{F}_{21}}{m_1} - \frac{\mathbf{F}_{12}}{m_2} = \frac{\mathbf{F}(\mathbf{r})}{m_1} + \frac{\mathbf{F}(\mathbf{r})}{m_2} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\mathbf{F}(\mathbf{r})$$

Define the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Then

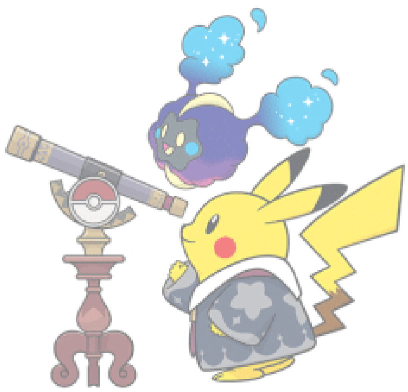
$$\mu\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r})$$

which is an effective one-body problem where a particle with mass μ moves under the force $\mathbf{F}(\mathbf{r})$. The angular momentum about the center of mass for the effective one-body problem with reduced mass μ is

$$\mathbf{L} = \mu\mathbf{r} \times \mathbf{v}$$

For a **central force**

$$\mathbf{F}(\mathbf{r}) = f(r)\hat{\mathbf{r}}$$



the torque is given by

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}(\mathbf{r}) = \mathbf{r} \times f(r)\hat{\mathbf{r}} = \mathbf{0}$$

Hence,

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau} = \mathbf{0} \quad \Longrightarrow \quad \mathbf{L} = \text{constant}$$

The total energy of the system is

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 + U(r)$$

where $U(r)$ is the potential energy of the central force. Note that

$$L = |\mathbf{L}| = \mu r^2\dot{\theta}$$

Using

$$\dot{\theta} = \frac{L}{\mu r^2}$$

we have

$$\frac{1}{2}\mu r^2\dot{\theta}^2 = \frac{L^2}{2\mu r^2}$$

$$E = \frac{1}{2}\mu\dot{r}^2 + \underbrace{\left[U(r) + \frac{L^2}{2\mu r^2}\right]}_{V_{\text{eff}}(r)}$$

7.1.4 n-body Problem

Consider N bodies with masses m_i and position vectors $\mathbf{r}_i(t)$ in three-dimensional space. According to Newton's law of gravitation, the acceleration of the i-th body is given by

$$\mathbf{r}_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} G\frac{m_j(\mathbf{r}_j - \mathbf{r}_i)}{\|\mathbf{r}_j - \mathbf{r}_i\|^3}, \quad i = 1, 2, \dots, N \tag{20}$$

where $\|\mathbf{r}_j - \mathbf{r}_i\|$ is the Euclidean distance between bodies i and j.

7.1.5 Binet's Equation

The classical Binet equation describes the shape of an orbit under a central gravitational force. It is given by

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{L^2}$$

where

- $u = \dfrac{1}{r}$ is the reciprocal of the radial distance,
- G is the gravitational constant,
- M is the mass of the central body, and
- L is the angular momentum per unit mass.

The general solution to this equation is an ellipse, with the form

$$u(\theta) = \frac{1}{r(\theta)} = \frac{GM}{L^2}\left(1 + e\cos(\theta - \theta_0)\right)$$

where

- e is the eccentricity of the orbit, and
- θ_0 is the initial phase of the orbit.

General Relativity introduces corrections to the orbit, particularly in the presence of a strong gravitational field. One important relativistic effect is the precession of the perihelion, which causes the orbit to shift over time. For orbits around a massive central body like the Sun, the general relativistic correction to the Binet equation can be written as

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{L^2}\left(1 + \frac{3GM}{c^2L^2}\right)$$

7.2 Kepler's Law

7.2.1 Three Kepler's Laws

Theorem. 7.4: Kepler's First Law

The orbit of every planet is an ellipse with the Sun at one of the two foci. The general equation for an ellipse in polar coordinates (r, θ) with one focus at the origin is

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta}$$

where a is the semi-major axis, and e is the eccentricity $(0 \leq e < 1)$. The eccentricity of an orbit describes how elliptical it is. For a perfectly circular orbit, the eccentricity is 0, while for an elliptical orbit, it is between 0 and 1. The closer the eccentricity is to 1, the more elongated the orbit.

1. Inner Solar System: The four terrestrial planets are located here, and their orbits are closer to the Sun with relatively small eccentricities.
2. Outer Solar System: The gas giants and icy bodies like comets have more elongated orbits and are located farther from the Sun.

Proof. Let the system be a planet of mass m orbiting a star of mass M (assumed fixed at the origin). The gravitational force is

$$\mathbf{F} = -G\frac{Mm}{r^2}\hat{r} = m\mathbf{a}$$

where $\mu = GM$ is the gravitational parameter and $\mathbf{r} = r\hat{r}$. The acceleration is $\mathbf{a} = -\frac{\mu}{r^2}\hat{r}$.

- **Angular Momentum L (Kepler's Second Law):** The torque is $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = 0$ (since $\mathbf{r}$ is parallel to $\mathbf{F}$). Therefore, the angular momentum is constant:

  $$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m(\mathbf{r} \times \mathbf{v}) = \text{constant}$$

  The orbit is confined to the plane perpendicular to $\mathbf{L}$.

- **Laplace-Runge-Lenz (LRL) Vector A:** The LRL vector is a constant of motion only for an inverse-square force. It is defined as

  $$\mathbf{A} = \mathbf{p} \times \mathbf{L} - m\mu\hat{r}$$

  To prove it's conserved, we show $\frac{d\mathbf{A}}{dt} = 0$. Using the product rule,

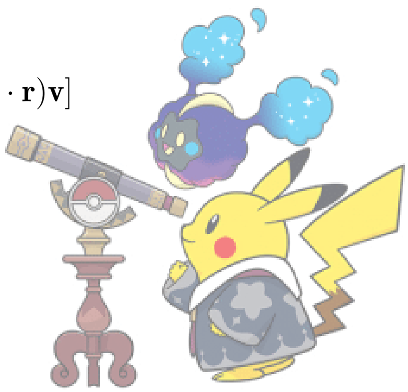
  $$\frac{d\mathbf{A}}{dt} = \frac{d\mathbf{p}}{dt} \times \mathbf{L} + \mathbf{p} \times \frac{d\mathbf{L}}{dt} - m\mu\frac{d\hat{r}}{dt}$$

  As $\frac{d\mathbf{p}}{dt} = \mathbf{F}$ and $\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau} = 0$,

  $$\frac{d\mathbf{A}}{dt} = \mathbf{F} \times \mathbf{L} - m\mu\frac{d\hat{r}}{dt}$$

  By $\mathbf{F} = -\frac{m\mu}{r^2}\hat{r}$ and $\mathbf{L} = m(\mathbf{r} \times \mathbf{v})$:

  $$\mathbf{F} \times \mathbf{L} = -\frac{m\mu}{r^2}\hat{r} \times m(\mathbf{r} \times \mathbf{v}) = -\frac{m^2\mu}{r^2}\left[(\hat{r}\cdot\mathbf{v})\mathbf{r} - (\hat{r}\cdot\mathbf{r})\mathbf{v}\right]$$



By $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$, $\hat{r} \cdot \mathbf{r} = r$ and $\hat{r} \cdot \mathbf{v} = \dot{r}$,

$$\mathbf{F} \times \mathbf{L} = -\frac{m^2\mu}{r^2}\left[(\dot{r})\mathbf{r} - (r)\mathbf{v}\right] = m\mu\left[\frac{r\mathbf{v}}{r^2} - \frac{\dot{r}\mathbf{r}}{r^2}\right]$$

Note that

$$\frac{d\hat{r}}{dt} = \frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) = \frac{\dot{\mathbf{r}}r - \mathbf{r}\dot{r}}{r^2} = \frac{\mathbf{v}r - \mathbf{r}\dot{r}}{r^2}$$

Hence, $\mathbf{F} \times \mathbf{L} = m\mu\frac{d\hat{r}}{dt}$. Therefore,

$$\frac{d\mathbf{A}}{dt} = m\mu\frac{d\hat{r}}{dt} - m\mu\frac{d\hat{r}}{dt} = 0$$

The LRL vector $\mathbf{A}$ lies in the orbital plane. We find the orbit equation by taking the dot product of $\mathbf{A}$ with the position vector $\mathbf{r}$:

$$\mathbf{r} \cdot \mathbf{A} = \mathbf{r} \cdot (\mathbf{p} \times \mathbf{L} - m\mu\hat{r}) = \mathbf{r} \cdot (\mathbf{p} \times \mathbf{L}) - m\mu(\mathbf{r} \cdot \hat{r})$$

Using the scalar triple product identity $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$,

$$\mathbf{r} \cdot (\mathbf{p} \times \mathbf{L}) = \mathbf{L} \cdot (\mathbf{r} \times \mathbf{p})$$

Since $\mathbf{r} \times \mathbf{p} = \mathbf{L}$, the identity simplifies to

$$\mathbf{r} \cdot (\mathbf{p} \times \mathbf{L}) = \mathbf{L} \cdot \mathbf{L} = L^2$$

Therefore,

$$\mathbf{r} \cdot \mathbf{A} = L^2 - m\mu r$$

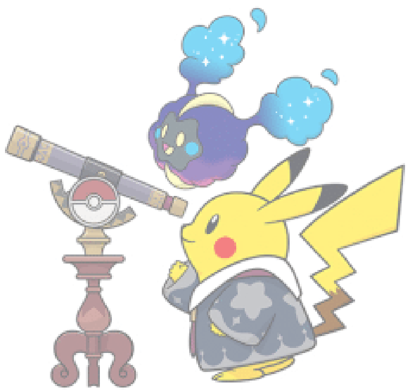
The LRL vector $\mathbf{A}$ is constant. Let $A = |\mathbf{A}|$. We define the angle θ between $\mathbf{r}$ and $\mathbf{A}$. Then $\mathbf{r} \cdot \mathbf{A} = rA\cos\theta$:

$$rA\cos\theta = L^2 - m\mu r$$

$$r = \frac{L^2}{m\mu}\frac{1}{1 + \frac{A}{m\mu}\cos\theta}$$

Normalizing by $m = 1$ gives

$$r = \frac{\ell}{1 + e\cos\theta}$$



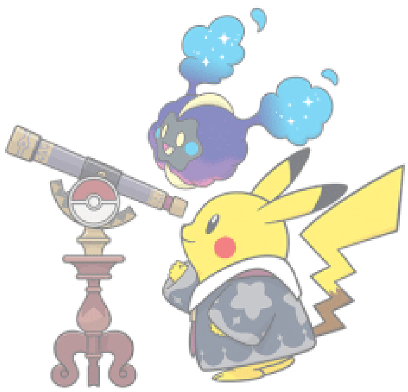
which is the equation of conic section in polar coordinates, where

- The **semi-latus rectum** is $\ell = \dfrac{L^2}{m\mu}$.
- The **eccentricity** is $e = \dfrac{A}{m\mu}$.

As A, m, and μ are all positive, the eccentricity e is a non-negative constant.

- If the total energy $E < 0$, then $0 \leq e < 1$, which defines an ellipse.
- If $E = 0$, then $e = 1$, defining a parabola.
- If $E > 0$, then $e > 1$, defining a hyperbola.

Planetary orbits are bound orbit, meaning the total mechanical energy E is negative ($E < 0$). This guarantees $0 \leq e < 1$, proving that the orbit is an ellipse with the Sun (the central mass) at one focus.



Theorem. 7.5: Kepler's Second Law

A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.

Proof.

- The gravitational force $\mathbf{F}$ is always directed along the line connecting the planet and the Sun (it's a **central force**).

$$\mathbf{F} = -G\frac{Mm}{r^2}\hat{r}$$

- The **torque** ($\boldsymbol{\tau}$) on the planet with respect to the Sun (the origin) is given by

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \left(-G\frac{Mm}{r^2}\hat{r}\right)$$

  Since $\mathbf{r}$ and $\hat{r}$ are in the same direction, $\mathbf{r} \times \hat{r} = 0$. Hence, the torque is zero: $\boldsymbol{\tau} = 0$.

- By Newton's second law for rotation, the torque is equal to the rate of change of the angular momentum ($\mathbf{L}$):

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$$

  Since $\boldsymbol{\tau} = 0$, the angular momentum $\mathbf{L}$ is conserved:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times (m\mathbf{v}) = \text{constant}$$

  where $\mathbf{p}$ is the linear momentum.

- The angular momentum can be expressed in terms of the area swept out per unit time. Consider an infinitesimal time dt. The area dA swept by the radius vector $\mathbf{r}$ is approximately the area of a triangle:

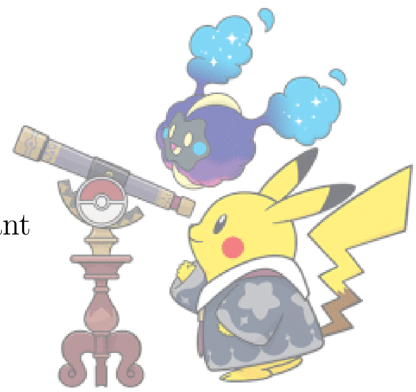
$$dA = \frac{1}{2}|\mathbf{r} \times d\mathbf{r}|$$

  Since $d\mathbf{r} = \mathbf{v}dt$, we have:

$$dA = \frac{1}{2}|\mathbf{r} \times \mathbf{v}dt| = \frac{1}{2}|\mathbf{r} \times \mathbf{v}|dt$$

- The rate at which area is swept out is $\dfrac{dA}{dt}$:

$$\frac{dA}{dt} = \frac{1}{2}|\mathbf{r} \times \mathbf{v}| = \frac{1}{2m}|\mathbf{r} \times m\mathbf{v}| = \frac{|\mathbf{L}|}{2m} = \text{constant}$$



Theorem. 7.6: Kepler's Third Law

Kepler's Third Law states that the square of the orbital period T of a planet is directly proportional to the cube of the semi-major axis a of its orbit. Mathematically,

$$T^2 \propto a^3$$

In the case of elliptical orbits, where a is the semi-major axis, the law still holds and it can be expressed as

$$T^2 = \frac{4\pi^2}{GM} a^3$$

where

- T is the orbital period of the planet.
- a is the semi-major axis of the ellipse.
- G is the gravitational constant.
- M is the mass of the central body (e.g., the Sun).

Proof. By Kepler's 2nd law,

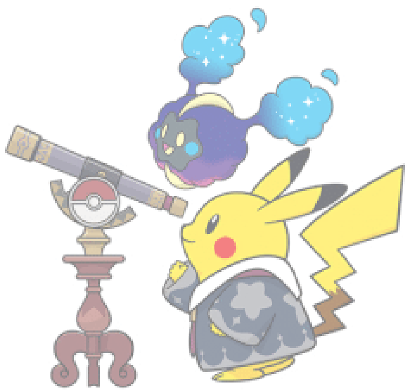
$$\frac{\pi ab}{T} = \frac{L}{2m}$$

as the area of ellipse is given by πab. By Kepler's 1st law, $r = a(1-e)$ and

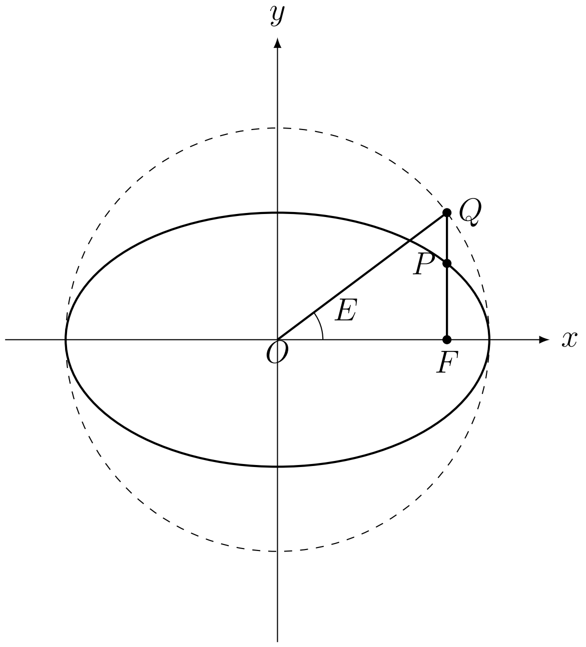
$$r(\theta) = \frac{L^2}{GMm^2(1 + e\cos\theta)}.$$

Then $\frac{L^2}{m^2} = a(1-e^2)GM$. Substitute the formula involving the area, $T^2 = \frac{4\pi^2}{GM} a^3$.

In some regions of the Solar System, objects have synchronized orbital periods due to gravitational interactions. This phenomenon, known as orbital resonance, is particularly important in the asteroid belt and among some moons of the giant planets.



7.2.2 Kepler's Equation



The eccentric anomaly E is defined using an auxiliary circle:

- Draw a circle of radius a centered at the origin (the auxiliary circle).
- For a point P on the ellipse, draw a vertical line to intersect the auxiliary circle at Q.
- The eccentric anomaly E is the angle $\angle XOQ$.

where O is the center of the ellipse and X is the positive x-axis direction. Although E is not a physical angle measured from the focus, it provides a convenient parametrization of the ellipse. Using the eccentric anomaly E, the position of the orbiting body is

$$\boxed{\begin{aligned} x &= a(\cos E - e) \\ y &= a\sqrt{1-e^2}\sin E \end{aligned}}$$

The **true anomaly** ν is the physical polar angle of the orbiting body measured from periapsis. The relation between ν and E is

$$\tan\frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$$

The area swept from periapsis to point P on the ellipse is

$$A = \frac{a^2}{2}(E - e\sin E)$$

The total area of the ellipse is $\pi a^2\sqrt{1-e^2}$, but by Kepler's second law, the fraction of area swept equals the fraction of time elapsed:

$$\frac{A}{\pi a^2} = \frac{t-\tau}{T}$$

where T is the orbital period. Using mean anomalty

$$M \equiv n(t-\tau)$$

(where **mean motion** $n = 2\pi/T$ and τ is the time of periapsis passage), we obtain

$$M = E - e\sin E$$

which is the Kepler's equation.

7.2.3 Poisson Bracket

Definition. 7.1: Poisson Bracket

The Poisson bracket of two functions $f(q,p)$ and $g(q,p)$, where q and p represent generalized coordinates and momenta, is defined as

$$\{f,g\} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right)$$

Here, q_i and p_i represent the generalized coordinates and momenta for the i-th degree of freedom.

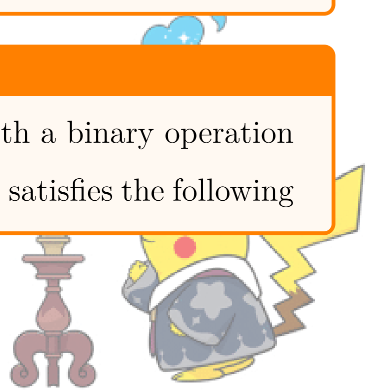
Definition. 7.2: Levi-Civita Symbol

The Levi-Civita symbol is defined as

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3), \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3), \\ 0 & \text{if any two indices are equal.} \end{cases}$$

Definition. 7.3: Lie Algebra

A **Lie algebra** is a vector space $\mathfrak{g}$ over a field (typically $\mathbb{R}$ or $\mathbb{C}$) equipped with a binary operation called the **Lie bracket**, denoted by $[\cdot,\cdot]$. The Lie bracket is a bilinear map that satisfies the following



two properties:

- **Bilinearity:**

$$[aX+bY,Z]=a[X,Z]+b[Y,Z],\quad [Z,aX+bY]=a[Z,X]+b[Z,Y]\quad \text{for all } X,Y,Z\in\mathfrak{g},\quad a,b\in\mathbb{F}$$

- **Antisymmetry:**

$$[X,Y]=-[Y,X]\quad \text{for all } X,Y\in\mathfrak{g}$$

- **Jacobi Identity:**

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0\quad \text{for all } X,Y,Z\in\mathfrak{g}$$

For the angular momentum $\mathbf{L}=\mathbf{r}\times\mathbf{p}$, the components of the angular momentum are

$$L_i=\epsilon_{ijk}r_jp_k$$

where ϵ_{ijk} is the Levi-Civita symbol, and we are summing over repeated indices.

$$\{p_i,L_j\}=\{p_i,\epsilon_{jkl}r_kp_l\}=\epsilon_{jkl}\left(\{p_i,r_kp_l\}\right)$$

As the position r_k and momentum p_l are conjugate variables, we have

$$\{p_i,r_k\}=-\delta_{ik}$$

and

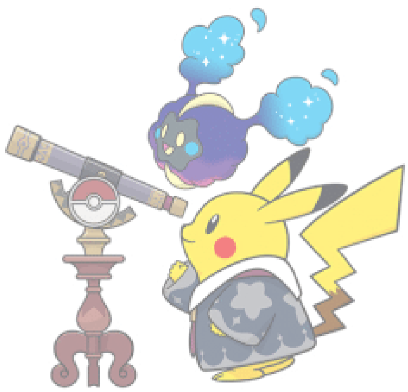
$$\{p_i,p_l\}=0$$

Note that

$$\{p_i,r_kp_l\}=r_k\{p_i,p_l\}+p_l\{p_i,r_k\}=-p_l\delta_{ik}$$

Finally,

$$\{p_i,L_j\}=\epsilon_{jkl}(-p_l\delta_{ik})=-\epsilon_{jkl}p_l\delta_{ik}=\epsilon_{ijk}p_k$$



Define $\mathbf{W} = \left\{\frac{\mathbf{p}}{m}, \mathbf{L}\right\} - \frac{GMm}{r}\mathbf{r}$ which is generalized LRL vector. One can prove that

$$\{W_i, W_j\} = \epsilon_{ijk}\left(\frac{-2E}{m}L_k\right)$$

and show that it forms lie algebra.

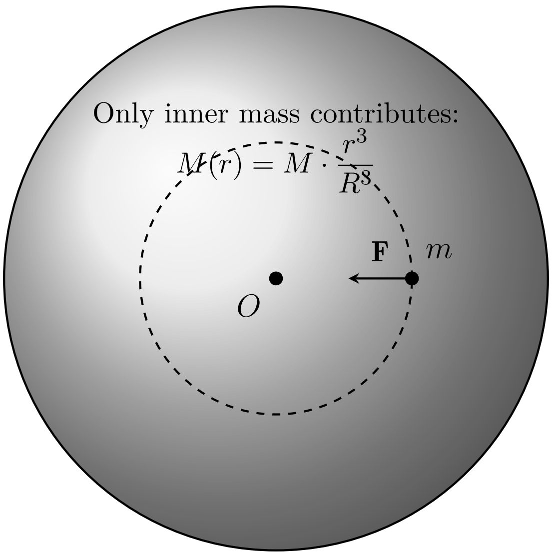
7.3 Shell Theorem

Theorem. 7.7: Shell Theorem

- A spherically symmetric shell of mass exerts **no net gravitational force** on a particle located **inside** the shell.
- A spherically symmetric shell of mass exerts a **gravitational force on an external point mass** as if all the mass were concentrated at the shell's center.

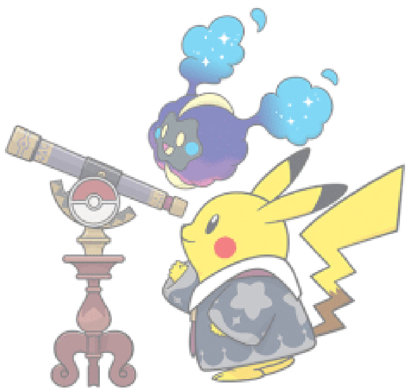
For a point mass located at a distance $r < R$ inside a uniform solid sphere (e.g., a planet), the Shell Theorem implies:

- Only the mass enclosed within radius r contributes to the gravitational force.
- Mass at radii greater than r exerts no net force by First Shell Theorem by considering it as infinitesimal shells.



Assuming uniform density, the mass enclosed within radius r is

$$M(r) = M \cdot \frac{r^3}{R^3}$$



Then, the gravitational force on a test mass m located at distance r from the center is

$$F = \frac{GM(r)m}{r^2} = \frac{GMm}{R^3} \cdot r$$

by second shell theorem.

Example. (Simple Gravity Train) Mass outside the spherical surface of radius r produces **zero net force**. The mass enclosed is

$$M(r) = M \cdot \frac{r^3}{R^3}$$

The gravitational force magnitude is therefore

$$F = -G\frac{M(r)m}{r^2} = -\frac{GMm}{R^3}r = m\ddot{r}$$

which is in Simple Harmonic Form (SHM) form $a = -\omega^2 x$. Therefore, $\omega = \sqrt{\frac{GM}{R^3}}$ and time of travel $t = \frac{\pi}{\sqrt{\frac{GM}{R^3}}} = \pi\sqrt{\frac{R}{g}}$.

Remark. The gravity train (also known as the Gauss gun, gravity elevator, or vacuum train) is a fascinating thought experiment in classical mechanics. First proposed by Robert Hooke in the 17th century and later analyzed by Isaac Newton, it explores what would happen if a tunnel were drilled through the Earth and an object were dropped through it.

Example. (Generalised Gravity Train) Consider a **straight tunnel** connecting two arbitrary points A and B on Earth's surface, not necessarily antipodal. The tunnel forms a **chord** of Earth's circular cross-section.

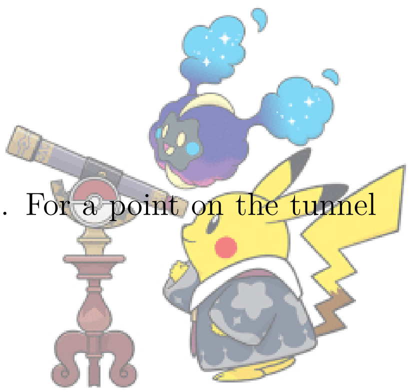
Let s be the coordinate along the tunnel, with $s = 0$ at the midpoint M. The endpoints are at $s = \pm L/2$. For a point **inside a homogeneous Earth** at position $\mathbf{r}$ from the center, the gravitational force is radial:

$$\mathbf{F} = -\frac{GM(r)}{r^2}\hat{\mathbf{r}}, \quad \text{with} \quad M(r) = \frac{r^3}{R^3}M$$

Hence,

$$\mathbf{F} = -\frac{GM}{R^3}r\,\hat{\mathbf{r}}$$

Let the tunnel be at a constant perpendicular distance y from Earth's center. For a point on the tunnel



at coordinate s, its radial distance from Earth's center is:

$$r(s) = \sqrt{s^2 + y^2}$$

The **component of gravitational force along the tunnel** (in the $\hat{\mathbf{s}}$ direction) is

$$F_s = \mathbf{F} \cdot \hat{\mathbf{s}} = \left(-\frac{GM}{R^3} r \, \hat{\mathbf{r}} \right) \cdot \hat{\mathbf{s}}$$

Since $\hat{\mathbf{r}} = \dfrac{s\hat{\mathbf{s}} + y\hat{\mathbf{y}}}{r}$, we have:

$$F_s = -\frac{GM}{R^3} r \left(\frac{s}{r} \right) = -\frac{GM}{R^3} s = m\ddot{s}$$

which is also in SHM form.

7.4 Gravitational Potential Energy

Definition. 7.4: Conservative Force

Potential energy can be defined only for **conservative forces**, which satisfy:

- The work done is path-independent.
- The net work over a closed loop is zero.
- A scalar potential function $U(\mathbf{r})$ exists.

Theorem. 7.8: Potential Energy

For a conservative force $\mathbf{F}$, the force is related to potential energy by

$$\mathbf{F} = -\nabla U$$

Theorem. 7.9: Gravitational Potential Energy

The gravitational potential energy (U) of an object in a gravitational field is defined as the work done in moving the object from infinity to a point at distance r from the source of the gravitational field. For two point masses, the gravitational potential energy is given by

$$U = -G\frac{m_1 m_2}{r}$$

where

- U is the gravitational potential energy,
- G is the gravitational constant,
- m_1 and m_2 are the two masses,
- r is the distance between the centers of the two masses.

Proof. The gravitational force is:

$$F = G\frac{Mm}{r^2}$$

Then,

$$\begin{aligned} U &= \int_{\infty}^{r} F\,dr \\ &= \int_{\infty}^{r} G\frac{Mm}{r^2}\,dr \\ &= GMm\int_{\infty}^{r} \frac{1}{r^2}\,dr \\ &= GMm\left[-\frac{1}{r}\right]_{\infty}^{r} \\ &= -G\frac{Mm}{r} \end{aligned}$$

7.4.1 Vis-viva Equation

Theorem. 7.10: Vis-viva Equation

The Vis-viva equation, also known as the orbital energy conservation equation, relates the velocity of an orbiting body to its position and the geometry of its orbit. It is expressed as

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

where

- v is the orbital speed of the orbiting body.
- r is the distance between the two bodies.
- a is the semi-major axis of the orbit.

- μ is the standard gravitational parameter (GM).

Proof. Let v_1 be the furthest distance from the object to the planet and v_2 be the closest distance from the object to the planet.

By conservation of angular momentum, $m_1 r_1 v_1 = m r_2 v_2 \implies v_2 = \frac{r_1}{r_2} v_1$. By conservation of mechanical energy $\frac{1}{2}mv^2 - \frac{GMm}{r} = \text{constant}$.

$$v_1^2 = \frac{2GMr_2}{r_1(r_1 + r_2)}$$

By $m_1 r_1 v_1 = m r_2 v_2 \implies v_2 = \frac{r_1}{r_2} v_1$ and $r_1 + r_2 = 2a$,

$$v^2 = \mu \left(\frac{2}{r} - \frac{1}{a} \right)$$

where G is the gravitational constant.

7.4.2 First Cosmic Speed, Second Cosmic Speed and Third Cosmic Speed

Definition. 7.5: First Cosmic Speed

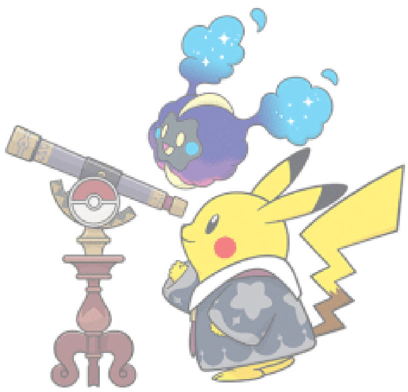
The first cosmic speed is the minimum velocity required for an object to orbit the Earth in a circular trajectory at the lowest possible altitude (negligible height above the surface).

For a satellite to remain in a circular orbit, the gravitational force must provide the required centripetal force.

$$F_{grav} = F_{centripetal}$$
$$\frac{GMm}{R^2} = m\frac{v_1^2}{R}$$

where

- M: Mass of Earth ($\approx 5.97 \times 10^{24}$ kg)
- m: Mass of the satellite
- R: Radius of Earth ($\approx 6.37 \times 10^{6}$ m)
- G: Gravitational Constant



$$v_1 = \sqrt{\frac{GM}{R}}$$

Definition. 7.6: Second Cosmic Speed

The second cosmic speed is the minimum velocity required for an object to break free from Earth's gravitational attraction, effectively escaping to infinity. It is commonly known as **escape velocity**.

The total energy at Earth's surface must equal the total energy at infinity (where both potential and kinetic energy are zero for a barely escape scenario).

$$K_{surface} + U_{surface} = K_{\infty} + U_{\infty}$$

$$\frac{1}{2}mv_2^2 - \frac{GMm}{R} = 0$$

$$v_2 = \sqrt{\frac{2GM}{R}}$$

Note that $v_2 = \sqrt{2}v_1$.

Definition. 7.7: Third Cosmic Speed

The third Cosmic Speed is the launch velocity required from Earth to escape the Solar System.

This requires overcoming Earth's gravity and then having enough residual velocity to escape the Sun's gravity.

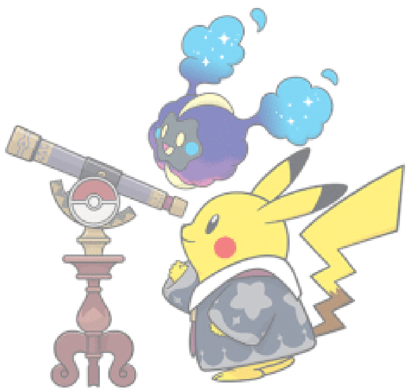
- **Hyperbolic Excess Velocity (v_{∞}):** Utilizing Earth's orbital speed ($v_E \approx 29.8\,\text{km/s}$), the spacecraft needs an excess velocity relative to Earth of

$$v_{\infty} = v_{esc,\odot} - v_E$$

- **Launch Calculation:** Applying energy conservation from Earth's surface:

$$\frac{1}{2}mv_3^2 - \frac{GMm}{R} = \frac{1}{2}mv_{\infty}^2$$

$$v_3 = \sqrt{v_{\infty}^2 + \frac{2GM}{R}}$$



Since $\frac{2GM}{R} = v_2^2$:

$$v_3 = \sqrt{v_\infty^2 + v_2^2}$$

7.4.3 Hohmann Transfer Orbit

The **Hohmann transfer orbit** is the most fuel-efficient two-impulse maneuver for transferring a spacecraft between two coplanar circular orbits around the same central body.

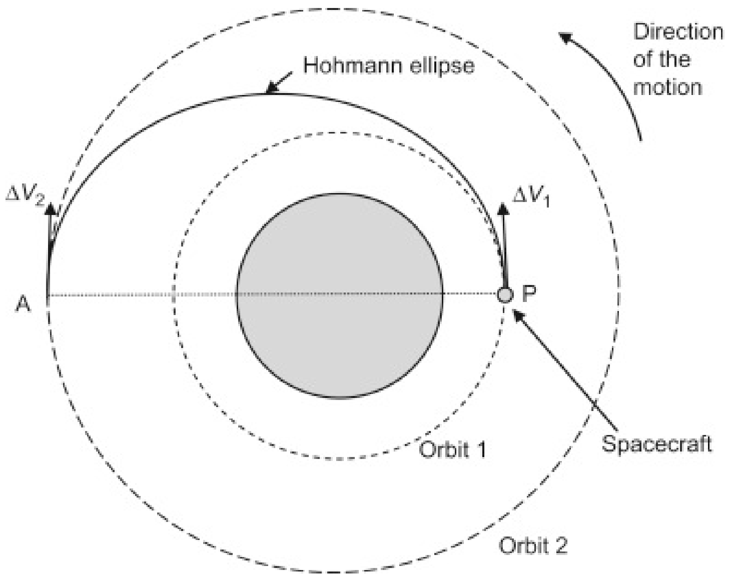


Figure 6: Source: https://www.sciencedirect.com/topics/engineering/hohmann-transfer

For a spacecraft in a circular orbit of radius r around a central body of mass M, the orbital speed is

$$v_c = \sqrt{\frac{GM}{r}}$$

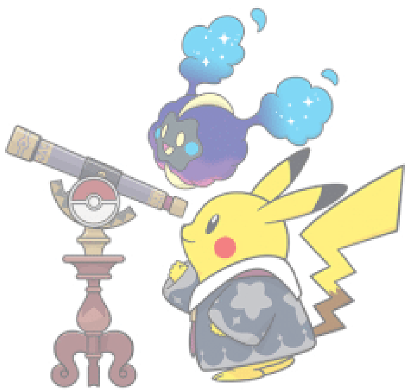
For any Keplerian orbit, the speed at distance r is given by the vis-viva equation:

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

where a is the semi-major axis of the orbit. Consider two circular orbits of radii r_1 (initial) and r_2 (final), with $r_2 > r_1$. The Hohmann transfer orbit is an ellipse with:

1. Periapsis at r_1,

2. Apoapsis at r_2.



The semi-major axis of the transfer ellipse is

$$a = \frac{r_1 + r_2}{2}$$

The spacecraft initially moves with circular speed

$$v_1 = \sqrt{\frac{GM}{r_1}}$$

At periapsis of the transfer ellipse, the required speed is obtained from the vis-viva equation:

$$v_p = \sqrt{GM\left(\frac{2}{r_1} - \frac{1}{a}\right)}$$

The first velocity increment is therefore

$$\Delta v_1 = v_p - v_1$$

This burn places the spacecraft onto the elliptical transfer orbit.

At apoapsis of the transfer ellipse ($r = r_2$), the spacecraft speed is

$$v_a = \sqrt{GM\left(\frac{2}{r_2} - \frac{1}{a}\right)}$$

The circular speed in the final orbit is

$$v_2 = \sqrt{\frac{GM}{r_2}}$$

The second velocity increment is

$$\Delta v_2 = v_2 - v_a$$

After this burn, the spacecraft is inserted into the final circular orbit.

The total fuel cost of the maneuver is measured by the total velocity increment:

$$\Delta v_{\text{total}} = \Delta v_1 + \Delta v_2$$

The Hohmann transfer minimizes Δv_{total} among all two-impulse transfers between coplanar circular orbits.

The spacecraft travels half of the elliptical orbit during the transfer. The orbital period of an ellipse is

$$T = 2\pi\sqrt{\frac{a^3}{GM}}$$

Hence, the Hohmann transfer time is

$$t_{\text{transfer}} = \pi\sqrt{\frac{a^3}{GM}}$$

This relatively long transfer time is the main trade-off for fuel efficiency.

7.4.4 Blackhole

The event horizon is a boundary in spacetime beyond which events cannot affect an observer. It is commonly associated with black holes, and marks the point at which the escape velocity exceeds the speed of light. This means that not even light can escape from inside the event horizon, which is why it appears black. For a non-rotating, spherically symmetric black hole, the event horizon is given by the Schwarzschild radius r_s, which is the radius at which the escape velocity equals the speed of light. The Schwarzschild radius is defined as

$$\boxed{r_s = \frac{2GM}{c^2}}$$

where

- G is the gravitational constant,
- M is the mass of the black hole,
- c is the speed of light.

Beyond the event horizon, all paths of particles and light curves back into the black hole, and no signal can escape to the outside universe.

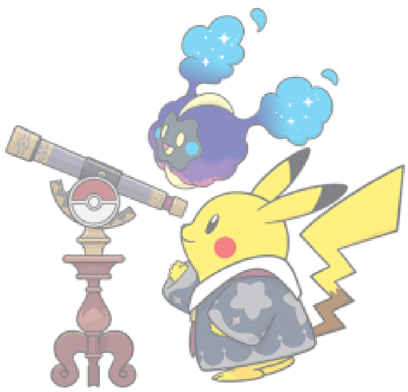
7.4.5 Virial Theorem

Homogeneous Function In the Virial Theorem, the **degree** refers to the degree of homogeneity of the potential energy function with respect to the spatial coordinates. A function $f(\mathbf{r})$ is said to be **homogeneous of degree** k if, for any scaling factor λ,

$$f(\lambda\mathbf{r}) = \lambda^k f(\mathbf{r})$$

Let the total potential energy of an N-particle system be

$$U(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$$



If U is homogeneous of degree k, then Euler's theorem for homogeneous functions gives

$$\sum_{i=1}^{N} \mathbf{r}_i \cdot \nabla_{\mathbf{r}_i} U = kU$$

Using the relation between force and potential,

$$\mathbf{F}_i = -\nabla_{\mathbf{r}_i} U$$

we obtain

$$\sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{r}_i = -kU$$

Virial Theorem Consider a bound system of N particles with total kinetic energy T and total potential energy U. If the system is stable and the potential energy is a homogeneous function of degree k in the coordinates, then the Virial Theorem states:

$$2\langle T\rangle = k\langle U\rangle$$

where $\langle\cdot\rangle$ denotes a time average.

Proof. Define the **virial** G as

$$G = \sum_{i=1}^{N} \mathbf{p}_i \cdot \mathbf{r}_i$$

where $\mathbf{p}_i = m_i \mathbf{v}_i$ is the momentum of the i-th particle. Taking the time derivative,

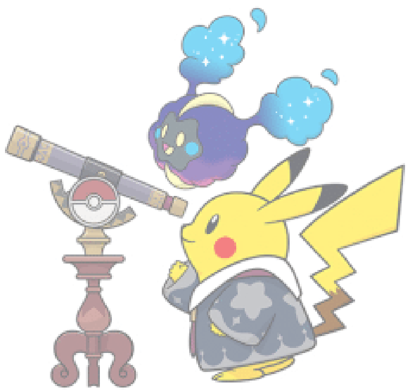
$$\frac{dG}{dt} = \sum_i \left(\frac{d\mathbf{p}_i}{dt} \cdot \mathbf{r}_i + \mathbf{p}_i \cdot \mathbf{v}_i \right)$$

Using Newton's second law $\dfrac{d\mathbf{p}_i}{dt} = \mathbf{F}_i$ and $\mathbf{p}_i \cdot \mathbf{v}_i = m_i v_i^2 = 2T_i$, we obtain

$$\frac{dG}{dt} = 2T + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i$$

For a bound system, G remains finite, so its time average satisfies

$$\left\langle \frac{dG}{dt} \right\rangle = 0$$



Hence,

$$2\langle T\rangle + \left\langle \sum_i \mathbf{F}_i \cdot \mathbf{r}_i \right\rangle = 0$$

If the forces derive from a potential U that is homogeneous of degree k, then

$$\sum_i \mathbf{F}_i \cdot \mathbf{r}_i = -kU$$

leading to

$$2\langle T\rangle = k\langle U\rangle$$

For gravitational potential,

$$U(r) \propto \frac{1}{r} \quad \Longrightarrow \quad U(\lambda r) = \lambda^{-1}U(r) \quad \Longrightarrow \quad k = -1$$

Hence,

$$2\langle T\rangle + \langle U\rangle = 0$$

Velocity Dispersion Consider a system of N particles with velocities $\mathbf{v}_i$. The mean velocity is

$$\langle \mathbf{v}\rangle = \frac{1}{N}\sum_{i=1}^{N} \mathbf{v}_i$$

The **velocity dispersion** is defined as the variance of the velocity distribution:

$$\sigma^2 = \left\langle |\mathbf{v} - \langle \mathbf{v}\rangle|^2 \right\rangle = \frac{1}{N}\sum_{i=1}^{N} |\mathbf{v}_i - \langle \mathbf{v}\rangle|^2$$

In many observations, only one velocity component (e.g. along the line of sight) is measurable. The one-dimensional velocity dispersion is

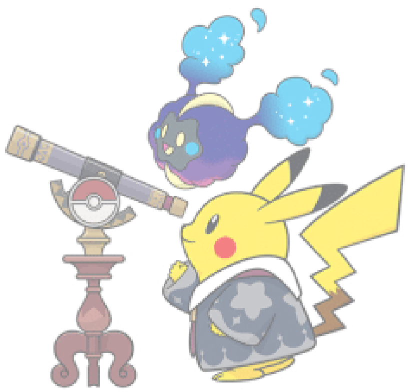
$$\sigma_{1\mathrm{D}}^2 = \left\langle (v_x - \langle v_x\rangle)^2 \right\rangle$$

For an isotropic system,

$$\sigma^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2 = 3\sigma_{1\mathrm{D}}^2$$

The total kinetic energy of a system of mass M is

$$T = \frac{1}{2}\sum_i m_i v_i^2$$



If the bulk motion has been subtracted and the velocity distribution is isotropic, then

$$T = \frac{1}{2}M\langle v^2 \rangle = \frac{3}{2}M\sigma_{1D}^2 = \frac{1}{2}M\sigma^2$$

For a gravitationally bound system in virial equilibrium,

$$2T + U = 0$$

Using $T \sim \frac{1}{2}M\sigma^2$ and $U \sim -\frac{GM^2}{R}$, we obtain

$$\sigma^2 \sim \frac{GM}{R}$$

Gravitational Potential Energy of a Uniform Solid sphere of Total Mass M and Radius R

The mass density is

$$\rho = \frac{M}{\frac{4}{3}\pi R^3}$$

The mass inside radius r is

$$M(r) = \frac{4}{3}\pi r^3 \rho$$

A thin spherical shell of radius r and thickness dr has mass

$$dM = 4\pi r^2 \rho \, dr$$

The gravitational potential energy gained by bringing the shell from infinity to radius r is

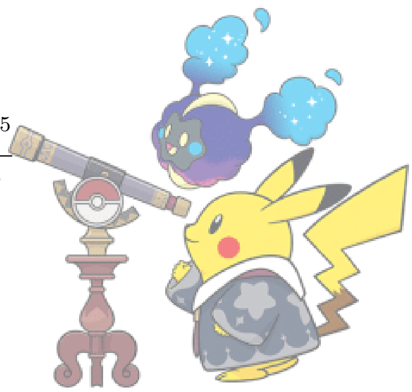
$$dU = -\frac{GM(r)\, dM}{r}$$

Substituting $M(r)$ and dM,

$$dU = -\frac{G}{r}\left(\frac{4}{3}\pi r^3 \rho\right)\left(4\pi r^2 \rho \, dr\right) = -\frac{16}{3}\pi^2 G \rho^2 r^4 \, dr$$

Integrate from $r = 0$ to $r = R$:

$$U = \int_0^R dU = -\frac{16}{3}\pi^2 G\rho^2 \int_0^R r^4 \, dr = -\frac{16}{3}\pi^2 G \rho^2 \frac{R^5}{5}$$



Thus,

$$U = -\frac{16}{15}\pi^2 G\rho^2 R^5$$

Substitute

$$\rho = \frac{3M}{4\pi R^3}, \qquad \rho^2 = \frac{9M^2}{16\pi^2 R^6}$$

Then,

$$U = -\frac{16}{15}\pi^2 G\left(\frac{9M^2}{16\pi^2 R^6}\right)R^5 = -\frac{3}{5}\frac{GM^2}{R}$$

It can provide order-of-magnitude estimates of gravitational potential energy of the uniform spherical galaxy model or other astrophysical objects. The coefficient $\frac{3}{5}$ assumes a **uniform density**. Real astrophysical objects are centrally concentrated, leading to different numerical factors. Nevertheless, the scaling

$$U \sim -\frac{GM^2}{R}$$

remains universally applicable.

7.4.6 Tully-Fisher Relation

In 1977, R. Brent Tully and J. Richard Fisher discovered an empirical relationship between the luminosity L of spiral galaxies and their maximum rotational velocity $v_{\max}$:

$$L \propto v_{\max}^{\alpha}$$

where observations give $\alpha \approx 3.5 - 4$ in the infrared and $\alpha \approx 2.5 - 3$ in the blue band.

7.5 Lagrange Point

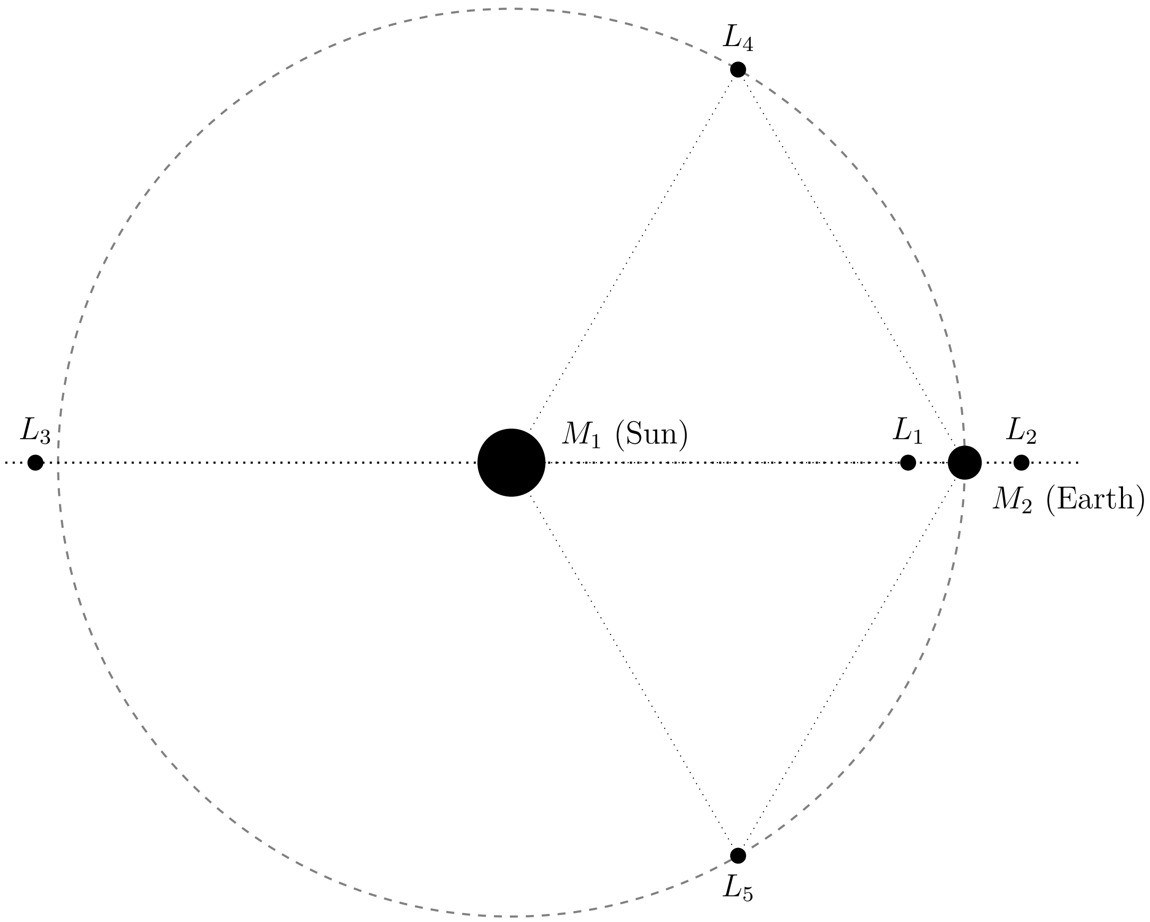


Figure 7: The Five Lagrangian Points (L_1 to L_5) in the Sun-Earth System's Rotating Frame

Two primary masses M_1 and M_2 orbit their common center of mass in circular motion. Let $M_1 > M_2$. Define the reduced mass

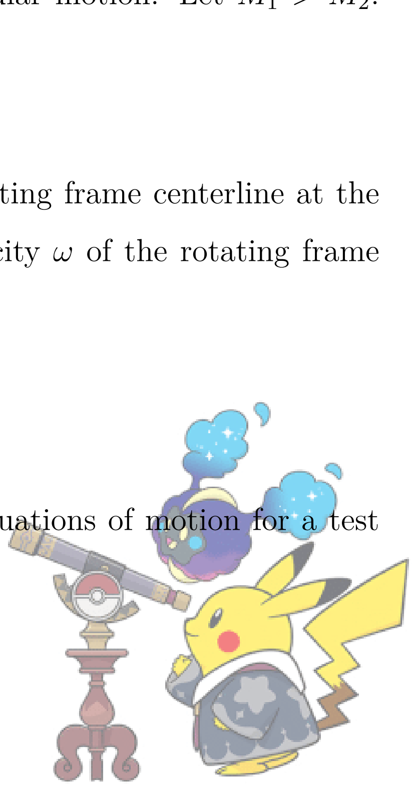
$$\mu = \frac{M_2}{M_1 + M_2}$$

Normalize $M_1 + M_2 = 1$, and the gravitational constant $G = 1$. In the rotating frame centerline at the center of mass, M_1 is at $(-\mu, 0)$ and M_2 is at $(1 - \mu, 0)$. The angular velocity ω of the rotating frame satisfies (from Kepler's 3rd law for M_1 and M_2)

$$\omega^2 = \frac{G(M_1 + M_2)}{a^3} = 1$$

since $a = 1$, $G = 1$, $M_1 + M_2 = 1$. So $\omega = 1$. In the rotating frame, the equations of motion for a test particle at $\mathbf{r} = (x, y)$ are

$$\ddot{x} - 2\omega\dot{y} = \frac{\partial U}{\partial x}, \quad \ddot{y} + 2\omega\dot{x} = \frac{\partial U}{\partial y}$$



where

$$U(x,y) = \underbrace{\frac{1}{2}\omega^2(x^2+y^2)}_{\text{Centrifugal Potential}} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

with

$$r_1 = \sqrt{(x+\mu)^2+y^2}, \quad r_2 = \sqrt{(x-1+\mu)^2+y^2}$$

Lagrange points are stationary points of U in the rotating frame (equilibrium points where $\dot{x} = \dot{y} = 0$):

$$\frac{\partial U}{\partial x} = 0, \quad \frac{\partial U}{\partial y} = 0$$

$$\frac{\partial U}{\partial x} = x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3} = 0$$

$$\frac{\partial U}{\partial y} = y\left(1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3}\right) = 0$$

For $y = 0$, we have $r_1 = |x+\mu|$, $r_2 = |x-1+\mu|$. The x-equation becomes

$$x - \frac{1-\mu}{(x+\mu)^2}\operatorname{sgn}(x+\mu) - \frac{\mu}{(x-1+\mu)^2}\operatorname{sgn}(x-1+\mu) = 0$$

- $x > 1-\mu$ (beyond M_2):
$$x - \frac{1-\mu}{(x+\mu)^2} - \frac{\mu}{(x-1+\mu)^2} = 0$$

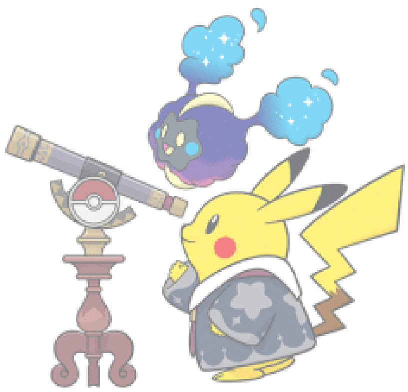
- $\mu < x < 1-\mu$ (between M_1 and M_2):
$$x - \frac{1-\mu}{(x+\mu)^2} + \frac{\mu}{(1-\mu-x)^2} = 0$$

- $x < -\mu$ (opposite side of M_1 from M_2):
$$x + \frac{1-\mu}{(x+\mu)^2} + \frac{\mu}{(x-1+\mu)^2} = 0$$

They have series solution:

$$\boxed{L_1: \quad x \approx 1-\mu - \left(\frac{\mu}{3}\right)^{1/3} + \frac{1}{3}\left(\frac{\mu}{3}\right)^{2/3} + \dots}$$

$$\boxed{L_2: \quad x \approx 1-\mu + \left(\frac{\mu}{3}\right)^{1/3} + \frac{1}{3}\left(\frac{\mu}{3}\right)^{2/3} + \dots}$$



$$\boxed{L_3: \quad x \approx -1 - \frac{5}{12}\mu + \dots}$$

From $\dfrac{\partial U}{\partial y} = 0$,

$$y\left(1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3}\right) = 0$$

If $y \neq 0$, then

$$\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3} = 1 \tag{*}$$

From $\dfrac{\partial U}{\partial x} = 0$,

$$x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3} = 0$$

Multiply (*) by x and subtract from above:

$$-\frac{(1-\mu)\mu}{r_1^3} + \frac{\mu(1-\mu)}{r_2^3} = 0 \quad \Longrightarrow \quad \frac{1-\mu}{r_1^3} = \frac{1-\mu}{r_2^3}$$

Hence, for $\mu \neq 1$, we get $r_1 = r_2$. From $r_1 = r_2$, (*) gives

$$\frac{1-\mu}{r_1^3} + \frac{\mu}{r_1^3} = \frac{1}{r_1^3} = 1 \quad \Longrightarrow \quad r_1 = 1$$

So $r_1 = r_2 = 1$. Now $(x+\mu)^2 + y^2 = 1$ and $(x-1+\mu)^2 + y^2 = 1$. Solving,

$$x = \frac{1}{2} - \mu, y = \pm\frac{\sqrt{3}}{2}$$

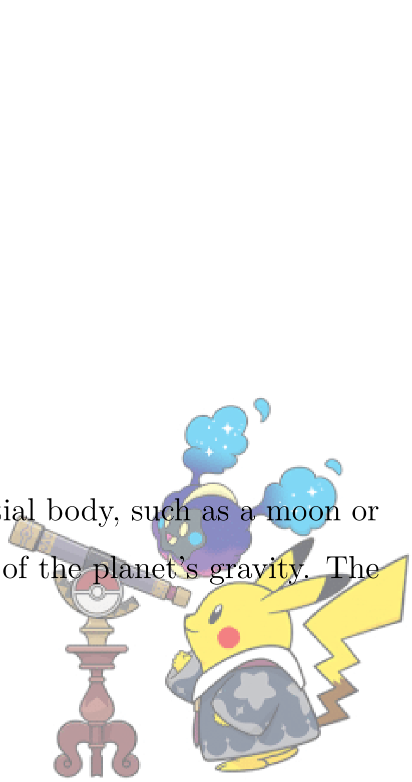
Therefore,

$$\boxed{L_4: \quad \left(\frac{1}{2} - \mu, \frac{\sqrt{3}}{2}\right)}$$

$$\boxed{L_5: \quad \left(\frac{1}{2} - \mu, -\frac{\sqrt{3}}{2}\right)}$$

7.6 Roche Limit

The Roche limit is the minimum distance from a planet within which a celestial body, such as a moon or asteroid, will experience tidal forces strong enough to break it apart because of the planet's gravity. The



Roche limit is given by the formula

$$\boxed{d_{\text{Roche}} = 2.44\, R \left(\frac{\rho_{\text{planet}}}{\rho_{\text{moon}}}\right)^{1/3}}$$

Derivation of simplified model: Tidal force is the difference in gravitational force felt across an object due to another massive body.

Consider a small object of radius R located at a distance d from a much more massive body of mass M.

$$a_{\text{near}} = \frac{GM}{(d-R)^2}, \quad a_{\text{far}} = \frac{GM}{(d+R)^2}$$

The tidal acceleration is the difference between the accelerations on the near and far sides

$$a_{\text{tidal}} = a_{\text{near}} - a_{\text{far}} = \frac{GM}{(d-R)^2} - \frac{GM}{(d+R)^2}$$

Assume $R \ll d$. Then

$$\frac{1}{(d \pm R)^2} \approx \frac{1}{d^2} \mp \frac{2R}{d^3}$$

Therefore,

$$a_{\text{tidal}} \approx GM\left(\left(\frac{1}{d^2} + \frac{2R}{d^3}\right) - \left(\frac{1}{d^2} - \frac{2R}{d^3}\right)\right) = GM \cdot \frac{4R}{d^3}$$

$$\boxed{a_{\text{tidal (per side)}} \approx \frac{2GMR}{d^3}}$$

Note that

$$a_{\text{tidal}} \approx \frac{2GM_pR_s}{d^3}, \quad a_{\text{self}} = \frac{Gm_s}{R_s^2}$$

By $m_s = \frac{4}{3}\pi R_s^3 \rho_s$ and $M_p = \frac{4}{3}\pi R_p^3 \rho_p$,

$$\frac{2GM_pR_s}{d^3} = \frac{Gm_s}{R_s^2}$$

which can be simplified to

$$\boxed{d = R_p \cdot \left(2 \cdot \frac{\rho_p}{\rho_s}\right)^{1/3}}$$

For the derivation of non-simplified model,

one can refer to https://dxwl.bnu.edu.cn/CN/abstract/abstract7639.shtml.

8 Relativity

Newtonian mechanics, formulated in the 17th century, works exceptionally well for everyday speeds ($v \ll c$). However, as experiments in the late 19th century became more precise (e.g., Michelson-Morley experiment), inconsistencies emerged that required a new theoretical framework.

8.1 Postulates of Special Relativity

- **Principle of Relativity:** The laws of physics are the same in all inertial reference frames.
- **Invariance of Light Speed:** The speed of light in vacuum, c, is the same in all inertial frames, regardless of the motion of the source or observer.

8.2 Galileo Transformation

In classical mechanics, the Galilean transformation relates the coordinates of an event as observed in two inertial frames of reference moving at a constant relative velocity. Suppose we have two inertial frames:

- Frame S: stationary frame with coordinates (x, y, z, t)
- Frame S': moving with constant velocity v along the x-axis relative to S, with coordinates (x', y', z', t')

The Galilean transformation equations are

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}$$

8.3 Lorentz Transformation

Definition. 8.1: Lorentz Factor

$$\gamma = \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \beta^2}}$$

where v is the speed of the object and $\beta := \dfrac{v}{c}$.

In special relativity, the Lorentz transformation relates the coordinates of an event as observed in two inertial frames moving at a constant velocity relative to each other. Unlike the Galilean transformation, it accounts for the constancy of the speed of light.

Theorem. 8.1: Lorentz Transformation

For two inertial frames S and S' with S' moving at velocity v along the x-axis relative to S, the Lorentz transformations are

$$\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

Theorem. 8.2: Inverse Lorentz Transformation

For two inertial frames S and S' with S' moving at velocity v along the x-axis relative to S, the inverse Lorentz transformations are

$$\begin{aligned} t &= \gamma\left(t' + \frac{vx'}{c^2}\right) \\ x &= \gamma(x' + vt') \\ y &= y' \\ z &= z' \end{aligned}$$

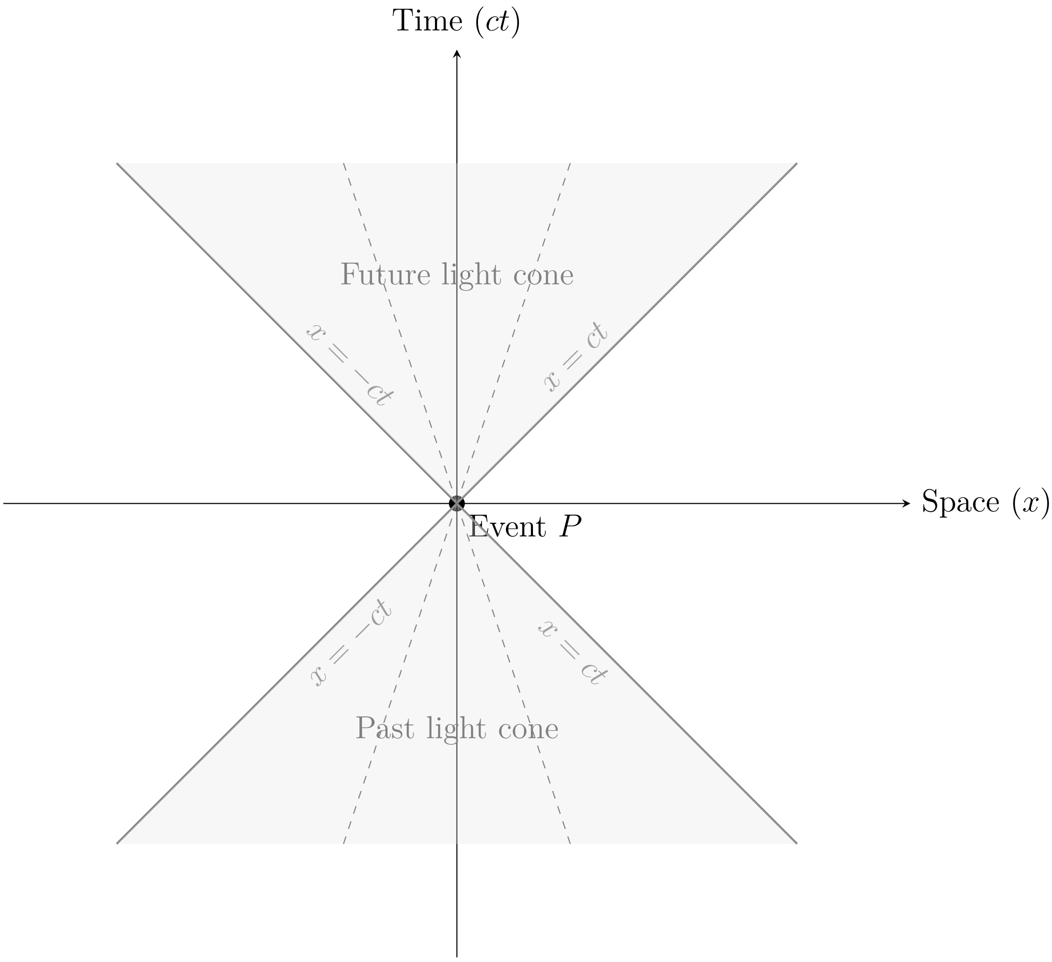
Theorem. 8.3: Lorentz Transformation for Velocity

For two inertial frames S and S' with S' moving at velocity v along the x-axis relative to S, the velocity transformations are

$$\begin{aligned} v'_x &= \frac{v_x - v}{1 - \frac{vv_x}{c^2}} \\ v'_y &= \frac{v_y}{\gamma\left(1 - \frac{vv_x}{c^2}\right)} \\ v'_z &= \frac{v_z}{\gamma\left(1 - \frac{vv_x}{c^2}\right)} \end{aligned}$$

8.4 Spacetime Diagram

A spacetime diagram is a graphical representation of events in spacetime.



8.5 Relativistic Kinematics and Mechanics

8.5.1 Time Dilation

Time dilation is one of the most fascinating consequences of special relativity. It states that time measured in a moving frame (proper time τ) appears to run slower compared to time measured in a stationary frame (coordinate time t). The relationship between proper time (τ) and coordinate time (t) is given by

$$\Delta t = \gamma \Delta \tau$$

Example. A spaceship travels at $v = 0.8c$ relative to Earth. If 5 years pass on the spaceship (proper time), how much time passes on Earth?

Solution.

$$\gamma = \frac{1}{\sqrt{1-(0.8)^2}} \approx 1.667$$

$$\Delta t = \gamma \Delta \tau = 1.667 \times 5 \approx 8.33 \text{ years}$$

8.5.2 Length Contraction

Length is defined as the distance between simultaneous measurements of an object's endpoints. For a rod of proper length L_0 at rest in S', moving with velocity v in S. In S, measure endpoints simultaneously ($\Delta t = 0$):

$$\begin{aligned}
\Delta x &= L \quad \text{(measurable length in S)} \\
\Delta x' &= L_0 \quad \text{(proper length in S')} \\
\Delta x' &= \gamma(\Delta x - v\Delta t) = \gamma L \\
L &= \frac{L_0}{\gamma}
\end{aligned}$$

Hence, moving objects appear shorter in their direction of motion.

8.5.3 Relativistic Momentum

Consider a particle with rest mass m_0.

$$\mathbf{p} = \gamma m_0 \mathbf{v}$$

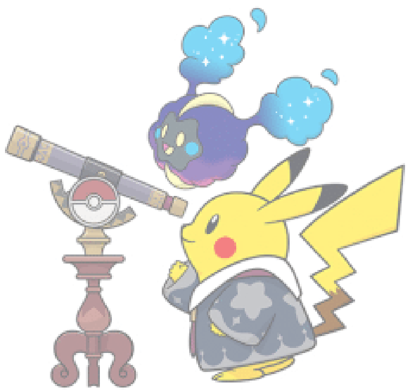
8.5.4 Relativistic Energy

The correct relativistic expression for kinetic energy is

$$K = (\gamma - 1)mc^2$$

Proof. Starting from the work-energy theorem and relativistic momentum:

$$\begin{aligned}
K &= \int \mathbf{F} \cdot d\mathbf{r} = \int \frac{d}{dt}(\gamma m_0 \mathbf{v}) \cdot \mathbf{v}\, dt \\
&= \int \left(\gamma m_0 \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} + m_0 \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} \frac{d\gamma}{dv^2} \frac{dv^2}{dt} \right) dt \\
&= m_0 c^2 \int d\gamma = \gamma m_0 c^2 + C
\end{aligned}$$



When $\gamma = 0$, $K = 0$. Therefore, $K = (\gamma - 1)m_0c^2$.

The total energy of a particle is

$$E = \gamma mc^2 = K + m_0c^2$$

This consists of two parts:

- **Rest energy:** $E_0 = m_0c^2$
- **Kinetic energy:** $K = (\gamma - 1)mc^2$

8.5.5 Relation between Relativistic Momentum and Energy

From our definitions:

$$E = \gamma m_0 c^2$$

$$\mathbf{p} = \gamma m_0 \mathbf{v}$$

Compute $E^2 - (pc)^2$:

$$\begin{aligned} E^2 - (pc)^2 &= (\gamma m_0 c^2)^2 - (\gamma m_0 vc)^2 \\ &= \gamma^2 m_0^2 c^4 \left(1 - \frac{v^2}{c^2}\right) \\ &= m_0^2 c^4 \frac{1 - \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}} \quad (\text{since } \gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}}) \\ &= m_0^2 c^4 \end{aligned}$$

Hence we obtain the fundamental relation:

$$\boxed{E^2 = (pc)^2 + (m_0c^2)^2}$$

8.6 Spacetime and Four Vector

8.6.1 Spacetime in Special Relativity

In special relativity, space and time are unified into a four-dimensional continuum called **spacetime**. Events are described by four coordinates:

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$$

where

- $x^0 = ct$ is the temporal coordinate (with c being the speed of light)
- x^1, x^2, x^3 are spatial coordinates

Example. A frame S', is moving with velocity v along the y-axis of another frame S. The Lorentz transformation in terms of $x^0 \equiv ct, x^1 \equiv x, x^2 \equiv y, x^3 \equiv z, x^0 \equiv ct'$ is

$$\begin{aligned} x'^0 &= \gamma(x^0 - \beta x^1) \\ x'^1 &= \gamma(x^1 - \beta x^0) \\ x'^2 &= x^2 \\ x'^3 &= x^3 \end{aligned}$$

where $\beta = \dfrac{v}{c}$ and $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$.

8.6.2 Four-Vector

A **four-vector** A^μ is a mathematical object that transforms under Lorentz transformations in the same way as spacetime coordinates:

$$A^\mu = (A^0, A^1, A^2, A^3)$$

The transformation rule for a Lorentz transformation Λ^μ_ν is

$$A'^\mu = \Lambda^\mu_\nu A^\nu$$

where

$$\Lambda^\mu_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Example. **(Continued)** A second-rank tensor has only one non-zero component in the S-frame, $T^{00} \equiv \rho$. The components in the S'-frame are obtained using the Lorentz transformation:

$$T'^{00} = \gamma^2 T^{00}, \quad T'^{01} = \gamma\beta T^{00}, \quad T'^{10} = \gamma\beta T^{00}, \quad T'^{11} = T^{00}$$

$$T'^{22} = T^{22}, \quad T'^{33} = T^{33}$$

Hence, the components of $T^{\mu\nu}$ in S'-frame are

$$T'^{00} = \gamma^2 \rho, \quad T'^{01} = T'^{10} = \gamma\beta\rho, \quad T'^{11} = \rho$$

8.6.3 Metric Tensor

The metric tensor $g_{\mu\nu}$ defines the geometry of spacetime. In special relativity with Cartesian coordinates, it takes a simple form known as the **Minkowski metric**:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

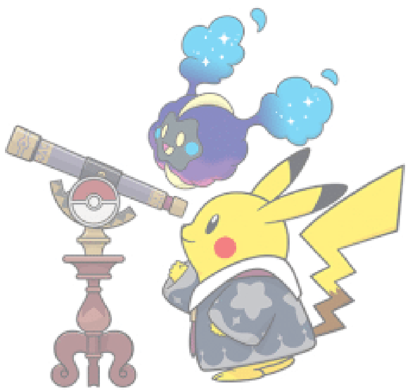
In component form:

$$g_{00} = 1, \quad g_{11} = g_{22} = g_{33} = -1, \quad g_{\mu\nu} = 0 \text{ for } \mu \neq \nu$$

The metric tensor allows conversion between contravariant (upper index) and covariant (lower index) vectors.

Lowering an index

$$A_\mu = g_{\mu\nu} A^\nu$$



Explicitly:

$$A_0 = g_{00}A^0 = A^0$$
$$A_1 = g_{11}A^1 = -A^1$$
$$A_2 = g_{22}A^2 = -A^2$$
$$A_3 = g_{33}A^3 = -A^3$$

Raising an index

$$A^\mu = g^{\mu\nu}A_\nu$$

where $g^{\mu\nu}$ is the inverse metric, which for Minkowski spacetime is identical to $g_{\mu\nu}$:

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The metric tensor and its inverse satisfy

$$g^{\mu\lambda}g_{\lambda\nu} = \delta^\mu_\nu = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases}$$

where δ^μ_ν is the Kronecker delta.

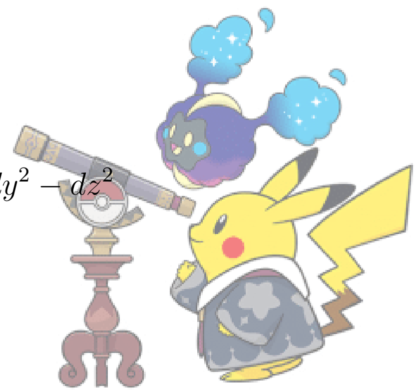
8.6.4 Invariant Interval

The **invariant interval** between two infinitesimally close events in spacetime is defined as:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

Expanding in components for Minkowski spacetime:

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$



For finite separation between events, the interval is:

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

The most important property of ds^2 is that it is **invariant** under Lorentz transformations. If we transform to a new coordinate system x'^μ:

$$ds'^2 = g_{\mu\nu} dx'^\mu dx'^\nu = g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma dx^\rho dx^\sigma = g_{\rho\sigma} dx^\rho dx^\sigma = ds^2$$

This invariance follows from the defining property of Lorentz transformations:

$$g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma}$$

8.6.5 Important Four-Vectors in Physics

Position Four-Vector

$$x^\mu = (ct, \mathbf{x}) = (ct, x, y, z)$$

Velocity Four-Vector The four-velocity is defined as the derivative of the position four-vector with respect to proper time τ:

$$u^\mu = \frac{dx^\mu}{d\tau} = (\gamma c, \gamma \mathbf{v})$$

where τ is proper time.

The components are

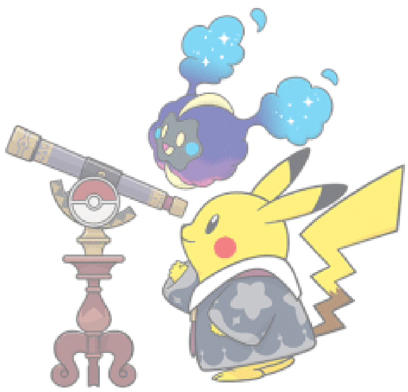
$$u^0 = \frac{dx^0}{d\tau} = \frac{c dt}{d\tau} = \gamma c$$
$$u^i = \frac{dx^i}{d\tau} = \frac{dx^i}{dt} \frac{dt}{d\tau} = \gamma v^i$$

Magnitude of four-velocity is given by

$$u^\mu u_\mu = \gamma^2 (c^2 - v^2) = \frac{c^2 - v^2}{1 - v^2/c^2} = c^2$$

Momentum Four-Vector

$$p^\mu = m u^\mu = \left(\frac{E}{c}, \mathbf{p} \right)$$



where $E = \gamma mc^2$ is relativistic energy and $\mathbf{p} = \gamma m\mathbf{v}$ is relativistic momentum.

Explicitly,

$$p^\mu = \left(\frac{E}{c}, p_x, p_y, p_z\right)$$

Furthermore,

$$p^\mu p_\mu = \frac{E^2}{c^2} - p^2 = m^2c^2$$

Example. Consider two spatial and one temporal dimensions. A particle emits a photon, making an angle θ' with the x'-axis in a frame (S'-frame). Hence, the trajectory of the photon can be written as $x' = ct'\cos\theta'$ and $y' = ct'\sin\theta'$.

(a) If the S'-frame is moving with velocity v with respect to another frame (S-frame), what is the trajectory of the photon in the S-frame?

(b) What is the energy-momentum four-vector of the photon in the S'-frame, if the frequency measured in this frame is f? (Of course, the z-component, p'_z, is zero.)

(c) Express the energy of the photon in the S-frame in terms of θ'.

(d) At which direction does the photon have maximum energy in the S-frame?

To find the trajectory in the S-frame, we apply the inverse Lorentz transformations. Assuming the S'-frame moves with velocity v along the x-axis of the S-frame, the coordinates transform as:

$$x = \gamma(x' + vt')$$

$$y = y'$$

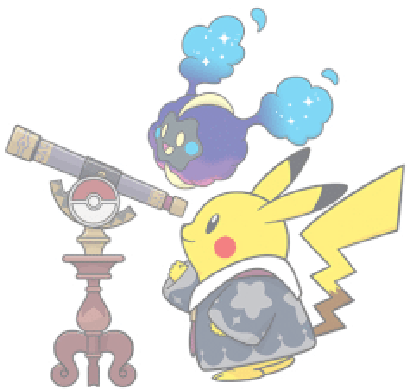
$$t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

where $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$ and $\beta = \dfrac{v}{c}$. Substituting the S'-frame trajectory $x' = ct'\cos\theta'$ and $y' = ct'\sin\theta'$:

(1) $x = \gamma(ct'\cos\theta' + vt') = \gamma t'(c\cos\theta' + v)$

(2) $y = ct'\sin\theta'$

(3) $t = \gamma\left(t' + \dfrac{v(ct'\cos\theta')}{c^2}\right) = \gamma t'(1 + \beta\cos\theta')$



To find $x(t)$ and $y(t)$, we solve for t' in terms of t from (3): $t' = \dfrac{t}{\gamma(1+\beta\cos\theta')}$. Substituting this into (1) and (2):

$$x(t) = \frac{c\cos\theta' + v}{1+\beta\cos\theta'}t, \quad y(t) = \frac{c\sin\theta'}{\gamma(1+\beta\cos\theta')}t$$

For a photon, $E' = hf$ and $p' = \dfrac{E'}{c} = \dfrac{hf}{c}$. The components of the momentum vector in the S'-frame are $p'_x = p'\cos\theta'$ and $p'_y = p'\sin\theta'$. The four-vector P'^{μ} is

$$P'^{\mu} = \left(\frac{E'}{c}, p'_x, p'_y, p'_z\right) = \left(\frac{hf}{c}, \frac{hf}{c}\cos\theta', \frac{hf}{c}\sin\theta', 0\right)$$

The energy E in the S-frame is obtained from the transformation $E = \gamma(E' + vp'_x)$:

$$E = \gamma\left(hf + v\frac{hf}{c}\cos\theta'\right) = \gamma hf(1+\beta\cos\theta')$$

The energy $E(\theta') = \gamma hf(1+\beta\cos\theta')$ is maximized when $\cos\theta'$ is at its maximum value. This occurs when $\cos\theta' = 1$, which corresponds to $\theta' = 0$. Therefore, the photon has maximum energy when emitted in the forward direction (the same direction as the velocity v).

8.7 Introduction to General Relativity

Motivation Classical Newtonian gravity describes gravitation as a force acting instantaneously at a distance. However, this framework is incompatible with special relativity, which asserts that

- The speed of light is finite and invariant.
- Space and time form a unified structure: spacetime.

General Relativity, formulated by Albert Einstein in 1915, resolves this incompatibility by replacing the concept of gravitational force with the **geometry of spacetime**.

Relevant Questions

Romanian Master of Physics 2014 Problem 3 **(Black Holes Physics)**, Romanian Master of Physics 2012 Problem 3 **(Fundaments of General Relativity)**

8.7.1 Operators in General Relativity

There are different operators in General Relativity. One of the examples is Christoffel symbols, which describe how coordinate axes change from point to point. In curved spacetime (or even flat space with

curved coordinates), basis vectors are not constant, taking an ordinary derivative of a vector does not give a tensor and we must correct for the "bending" of coordinates.

Definition. 8.2: Christoffel Symbols

Consider a coordinate basis $\{e_\mu\}$. Unlike Cartesian coordinates, these basis vectors generally depend on position:

$$\partial_\nu e_\mu \neq 0$$

We define the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ by

$$\partial_\nu e_\mu = \Gamma^\lambda_{\mu\nu} e_\lambda$$

Theorem. 8.4: Christoffel Symbols

The Christoffel symbols are defined as

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right)$$

where $\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu}$ is called the 4-gradient.

Another example is d'Alembertian operator.

Definition. 8.3: d'Alembertian Operator

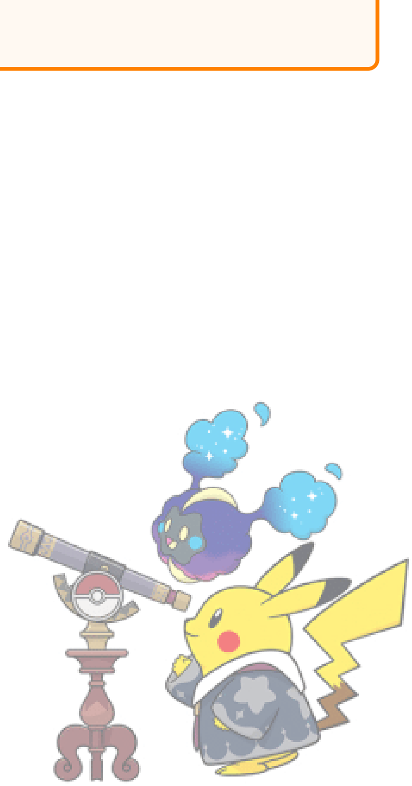
$$\Box = \partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

A scalar field $\phi(x^\mu)$ satisfies the relativistic wave equation

$$\Box \phi = 0$$

This equation can describe

- Electromagnetic waves
- Gravitational waves
- Massless scalar fields



For a scalar field of mass m, the equation becomes

$$\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\phi = 0$$

This is the relativistic generalization of the Schrödinger equation.

In general relativity, spacetime is curved, and partial derivatives must be replaced by **covariant derivatives**. The d'Alembertian becomes the **Laplace–Beltrami operator**:

$$\Box\phi = \nabla^\mu\nabla_\mu\phi$$

In coordinates, this can be written as

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi\right)$$

where $g = \det(g_{\mu\nu})$.

8.7.2 Introduction to Spacetime and Differentiable Manifold

In classical physics, space and time are treated as separate entities:

- Space: a three-dimensional Euclidean arena
- Time: a universal parameter flowing identically for all observers

However, experiments involving high velocities and electromagnetic waves reveal that:

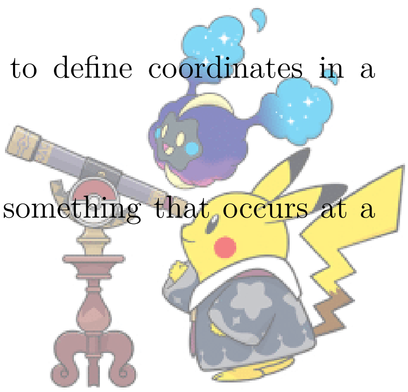
- The speed of light is invariant
- Measurements of time and length depend on the observer

These facts force a radical conclusion:

> Space and time are not independent; they form a unified structure called spacetime.

In the curved spacetime, we model spacetime as a **differentiable manifold** M:

- A manifold is a set of points that locally resembles $\mathbb{R}^n$, allowing us to define coordinates in a neighborhood of each point.
- For spacetime, $n = 4$ and each point represents an event. An event is something that occurs at a specific place and time.



- Smooth functions and tensors can be defined on the manifold, enabling calculus on curved spacetime.

Example. A sphere in $\mathbb{R}^n$ is a manifold.

Proof. We introduce some mathematical concepts first:

Definition. 8.4: Stereographic Projection

Stereographic projection is a mapping that projects a sphere onto a plane. Specifically, let $S^n \subset \mathbb{R}^{n+1}$ be the n-dimensional unit sphere, and let $N = (0, \ldots, 0, 1)$ be the north pole. The stereographic projection $\pi : S^n \setminus \{N\} \to \mathbb{R}^n$ is defined by mapping a point P on S^n (excluding N) to the intersection of the line through N and P with the hyperplane $x_{n+1} = 0$:

$$\pi(P) = \frac{1}{1 - x_{n+1}}(x_1, x_2, \ldots, x_n)$$

where $P = (x_1, x_2, \ldots, x_{n+1}) \in S^n$.

Definition. 8.5: Stereographic Projection

A **topological space** is a pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ is a collection of subsets of X (called open sets) satisfying:

1. $\varnothing, X \in \mathcal{T}$,
2. Any union of elements of $\mathcal{T}$ is in $\mathcal{T}$,
3. Any finite intersection of elements of $\mathcal{T}$ is in $\mathcal{T}$.

The collection $\mathcal{T}$ is called a topology on X.

Definition. 8.6: Homomorphism

Let X and Y be topological spaces. A function

$$f : X \to Y$$

is called a **homeomorphism** if

1. f is a **bijection** (one-to-one and onto),
2. f is **continuous**,

3. the inverse function $f^{-1} : Y \to X$ is also **continuous**.

If such a map exists, we say X and Y are **homeomorphic**, meaning they are **topologically equivalent**.

Theorem. 8.5: Criteria of a Smooth Manifold

A smooth manifold is a space that is locally **homeomorphic** to $\mathbb{R}^n$ and admits smooth transitions between overlapping charts. Demonstrating that the stereographic projection is a homeomorphism ensures that each point on S^n possesses a neighborhood that topologically resembles $\mathbb{R}^n$, which is important in establishing that S^n is a smooth manifold.

Let

$$N = (0, \ldots, 0, 1), \quad S = (0, \ldots, 0, -1)$$

be the north and south poles of S^n. Define the stereographic projection from the north pole:

$$\pi_N : S^n \setminus \{N\} \to \mathbb{R}^n$$

by

$$\pi_N(x_1, \ldots, x_{n+1}) = \left(\frac{x_1}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}} \right)$$

As the inverse map is given by

$$\pi_N^{-1}(y_1, \ldots, y_n) = \left(\frac{2y_1}{\|y\|^2 + 1}, \ldots, \frac{2y_n}{\|y\|^2 + 1}, \frac{\|y\|^2 - 1}{\|y\|^2 + 1} \right)$$

and both π_N and π_N^{-1} are continuous, π_N is a homeomorphism. Similarly, define

$$\pi_S : S^n \setminus \{S\} \to \mathbb{R}^n$$

by

$$\pi_S(x_1, \ldots, x_{n+1}) = \left(\frac{x_1}{1 + x_{n+1}}, \ldots, \frac{x_n}{1 + x_{n+1}} \right)$$

The two charts

$$(S^n \setminus \{N\}, \pi_N), \quad (S^n \setminus \{S\}, \pi_S)$$

cover S^n. Each component is a rational function of $y_1, \ldots, y_n$ with denominator $\|y\|^2$, which is nonzero

on the domain. Hence the transition map $\pi_S \circ \pi_N^{-1}$ is smooth. The stereographic charts provide a smooth atlas covering S^n, and the transition functions are smooth. Therefore, S^n satisfies all the requirements of a smooth manifold.

8.7.3 Metric Tensor

On this 4-dimensional manifold $\mathcal{M}$, the geometry is described by a metric tensor $g_{\mu\nu}(x)$, a smooth, symmetric, non-degenerate tensor field:

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$$

where the Einstein's notation uses Lorentzian signature $(-,+,+,+)$.

Example. In inertial coordinates (t, x, y, z),

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

which has **Lorentzian signature** $(-,+,+,+)$.

Locally, at any point $p \in \mathcal{M}$, there exists a coordinate system in which the metric reduces to the Minkowski form $\eta_{\mu\nu}$ and its first derivatives vanish, reflecting the **equivalence principle** which states that

> Locally, the effects of gravity are indistinguishable from the effects of acceleration.

8.7.4 Geodesics

In General Relativity, free-falling particles follow the **straightest possible paths** in curved spacetime, called **geodesics**:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0$$

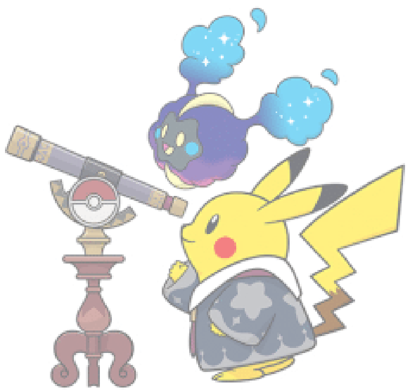
In a flat space (Euclidean geometry), the shortest path between two points is a straight line. In a curved space (like the surface of the Earth), the shortest path is a geodesic.

For timelike geodesics, the proper time τ is the affine parameter λ. For null (lightlike) geodesics, there is no proper time, but an affine parameter still exists to describe the geodesic.

8.7.5 Riemann Curvature Tensor

Curvature is measured by the Riemann tensor:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$



Contracting indices yields

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}$$

The Ricci scalar is

$$R = g^{\mu\nu} R_{\mu\nu}$$

8.7.6 Einstein's Field Equation

The field equations can be written as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

where

- $T_{\mu\nu}$ is the stress-energy tensor which describes the density and flux of energy and momentum.

In the weak-field, slow-motion limit:

$$g_{00} \approx -\left(1 + \frac{2\Phi}{c^2}\right)$$

Einstein's equations reduce to Poisson's equation:

$$\nabla^2 \Phi = 4\pi G \rho$$

Hence, Newtonian gravity emerges as an approximation of General Relativity.

8.7.7 Solution to Einstein's Field Equation

Schwarzschild solution Einstein's field equations are

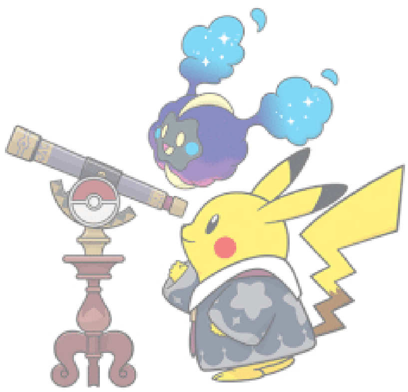
$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Outside a spherically symmetric gravitating body (e.g. outside a star or black hole), there is no matter or energy:

$$T_{\mu\nu} = 0$$

Hence the field equations reduce to the **vacuum Einstein equations**:

$$G_{\mu\nu} = 0$$



The most general line element consistent with these symmetries is

$$ds^2 = -e^{2\Phi(r)}c^2dt^2 + e^{2\Lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

where $\Phi(r)$ and $\Lambda(r)$ are unknown functions. Computing the Einstein tensor components for this metric and imposing $G_{\mu\nu} = 0$, one obtains the differential equations:

$$\frac{d}{dr}\left(r(1 - e^{-2\Lambda})\right) = 0$$
$$\frac{d\Phi}{dr} = \frac{1}{2r}\left(e^{2\Lambda} - 1\right)$$

Integrating the first equation gives

$$e^{-2\Lambda(r)} = 1 - \frac{C}{r}$$

where C is a constant of integration. Substituting this into the second equation and integrating yields

$$e^{2\Phi(r)} = 1 - \frac{C}{r}$$

Thus the spacetime metric becomes

$$ds^2 = -\left(1 - \frac{C}{r}\right)c^2dt^2 + \left(1 - \frac{C}{r}\right)^{-1}dr^2 + r^2d\Omega^2$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$. To determine C, we require that the metric reduces to Newtonian gravity in the weak-field limit $r \gg C$. In Newtonian gravity, the gravitational potential of a mass M is

$$\Phi_N(r) = -\frac{GM}{r}$$

In General Relativity, the weak-field limit gives

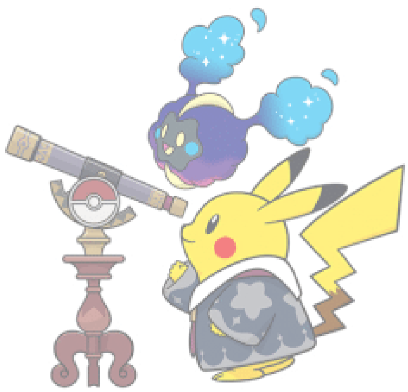
$$g_{tt} \approx -\left(1 + \frac{2\Phi_N}{c^2}\right)$$

Comparing with

$$g_{tt} = -\left(1 - \frac{C}{r}\right)$$

we identify

$$\frac{C}{r} = \frac{2GM}{c^2r}$$



Hence,

$$C = \frac{2GM}{c^2}$$

Substituting $C = \dfrac{2GM}{c^2}$, the metric becomes

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

This is the **Schwarzschild metric**. The event horizon occurs where the radial component of the metric diverges:

$$g_{rr}^{-1} = 1 - \frac{2GM}{c^2 r} = 0$$

Solving,

$$r = \frac{2GM}{c^2}$$

which is the **Schwarzschild radius**.

Friedmann–Lemaître–Robertson–Walker Metric It models an expanding universe:

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right]$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$.

Define χ such that

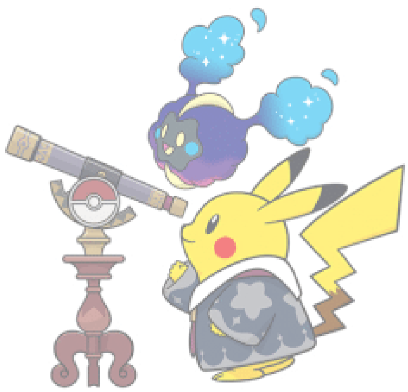
$$d\chi = \frac{dr}{\sqrt{1 - kr^2}}$$

Integrating gives the relations

$$r = \begin{cases} \dfrac{1}{\sqrt{k}} \sin\left(\sqrt{k}\chi\right) & , k > 1 \\ \chi & , k = 0 \\ \dfrac{1}{\sqrt{-k}} \sinh\left(\sqrt{-k}\chi\right) & , k < -1 \end{cases} = \begin{cases} \sin(\chi) & , k = 1 \\ \chi & , k = 0 \\ \sinh(\chi) & , k = -1 \end{cases}$$

as $k = -1, 0, 1$ only. The metric then becomes

$$ds^2 = -c^2 dt^2 + a^2(t) \left[d\chi^2 + S_k^2(\chi)\, d\Omega^2\right]$$



By natural unit $c = 1$,

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \dfrac{a^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & a^2r^2 & 0 \\ 0 & 0 & 0 & a^2r^2\sin^2\theta \end{pmatrix}$$

Then

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \dfrac{1-kr^2}{a^2} & 0 & 0 \\ 0 & 0 & \dfrac{1}{a^2r^2} & 0 \\ 0 & 0 & 0 & \dfrac{1}{a^2r^2\sin^2\theta} \end{pmatrix}$$

The non-zero Christoffel symbols are

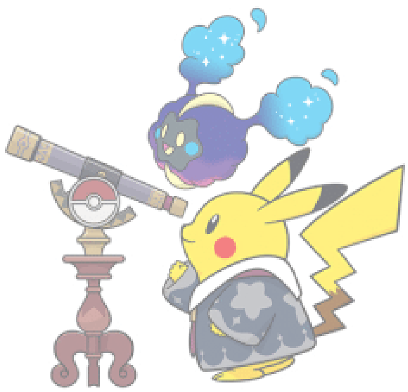
$$\begin{aligned} &\Gamma^0_{ij} = a\dot{a}\tilde{g}_{ij} = \frac{\dot{a}}{a}g_{ij} \\ &\Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j \\ &\Gamma^r_{rr} = \frac{kr}{1-kr^2}, \quad \Gamma^r_{\theta\theta} = -r(1-kr^2), \quad \Gamma^r_{\phi\phi} = -r(1-kr^2)\sin^2\theta \\ &\Gamma^\theta_{r\theta} = \Gamma^\phi_{r\phi} = \frac{1}{r} \\ &\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^\phi_{\theta\phi} = \cot\theta \end{aligned}$$

Then

$$\begin{aligned} R_{00} &= -3\frac{\ddot{a}}{a} \\ R_{ij} &= \left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2k}{a^2}\right]g_{ij} \\ R = g^{\mu\nu}R_{\mu\nu} &= 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right] \end{aligned}$$

For a perfect fluid in comoving coordinates,

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}$$



with $U^\mu = (1, 0, 0, 0)$, so:

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & pg_{11} & 0 & 0 \\ 0 & 0 & pg_{22} & 0 \\ 0 & 0 & 0 & pg_{33} \end{pmatrix}$$

By $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$,

$$\begin{aligned} -3\frac{\ddot{a}}{a} - \frac{1}{2}(-1) \cdot 6 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} \right] - \Lambda &= 8\pi G\rho \\ -3\frac{\ddot{a}}{a} + 3 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} \right] - \Lambda &= 8\pi G\rho \\ 3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3k}{a^2} - \Lambda &= 8\pi G\rho \\ \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \end{aligned}$$

This is the **First Friedmann Equation**.

$$\begin{aligned} \left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2k}{a^2} \right] g_{ii} - \frac{1}{2} g_{ii} \cdot 6 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} \right] + \Lambda g_{ii} &= 8\pi G p g_{ii} \\ \frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2k}{a^2} - 3 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} \right] + \Lambda &= 8\pi G p \\ -2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 - \frac{k}{a^2} + \Lambda &= 8\pi G p \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \end{aligned}$$

This is the **Second Friedmann Equation**.

8.7.8 Covariant Derivative

Let V^μ be a vector field. Under a coordinate transformation $x^\mu \to x^{\mu'}$, the partial derivative

$$\partial_\nu V^\mu$$

does not transform as a tensor because it produces extra terms involving second derivatives of the coordinate transformation. Hence, $\partial_\nu V^\mu$ has no invariant geometric meaning. The **covariant derivative** of a vector field V^μ is defined as

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\nu\lambda} V^\lambda$$

For a covector A_μ,

$$\nabla_\nu A_\mu = \partial_\nu A_\mu - \Gamma^\lambda_{\nu\mu} A_\lambda$$

More generally, for a (k, l) tensor,

$$\nabla_\rho T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} = \partial_\rho T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} + \sum_{i=1}^{k} \Gamma^{\mu_i}_{\rho\lambda} T^{\mu_1\cdots\lambda\cdots\mu_k}{}_{\nu_1\cdots\nu_l} - \sum_{j=1}^{l} \Gamma^{\lambda}_{\rho\nu_j} T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\lambda\cdots\nu_l}$$

In General Relativity, the covariant derivative is uniquely determined by two conditions:

1. **Metric compatibility:**

$$\nabla_\lambda g_{\mu\nu} = 0$$

2. **Torsion-free:**

$$\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$$

These conditions yield the **Levi–Civita connection**:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \tag{21}$$

The covariant derivative measures how a vector changes **relative to the curvature of the manifold**. It represents the best possible generalization of directional derivatives to curved spaces.

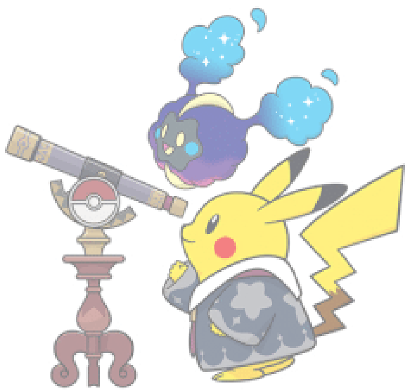
8.7.9 Lie Derivative

Let ξ^μ be a smooth vector field on a manifold $\mathcal{M}$. It generates a one-parameter family of diffeomorphisms (a **flow**)

$$\phi_\lambda : \mathcal{M} \to \mathcal{M}$$

satisfying

$$\frac{d}{d\lambda}\phi_\lambda(p) = \xi\big(\phi_\lambda(p)\big), \quad \phi_0(p) = p$$



For a scalar field f, the Lie derivative along ξ^μ is simply

$$\mathcal{L}_\xi f = \xi^\mu \partial_\mu f$$

which coincides with the directional derivative.

Let V^μ be a vector field. The Lie derivative of V^μ along ξ^μ is defined as

$$\mathcal{L}_\xi V^\mu = \xi^\nu \partial_\nu V^\mu - V^\nu \partial_\nu \xi^\mu \tag{22}$$

This expression measures the failure of V^μ to be invariant under the flow generated by ξ^μ. Equation (22) can be written compactly as

$$\mathcal{L}_\xi V = [\xi, V]$$

where $[\xi, V]$ is the **Lie bracket**:

$$[\xi, V]^\mu = \xi^\nu \partial_\nu V^\mu - V^\nu \partial_\nu \xi^\mu \tag{23}$$

Hence, the Lie derivative of a vector field is the Lie bracket.

8.7.10 Killing Vector Field

A vector field ξ^μ is called a **Killing vector field** if the metric remains invariant under the flow generated by ξ^μ:

$$\mathcal{L}_\xi g_{\mu\nu} = 0 \tag{24}$$

where $\mathcal{L}_\xi$ denotes the Lie derivative. Using properties of the Lie derivative, the Killing condition is equivalent to

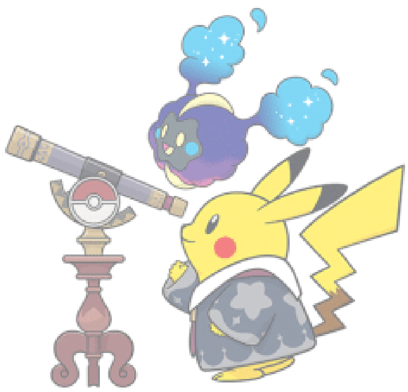
$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \tag{25}$$

This is known as the **Killing equation**. Equation (25) states that the symmetrized covariant derivative of ξ^μ vanishes. Hence, infinitesimal displacements along ξ^μ preserve the spacetime interval:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

In flat spacetime with metric $\eta_{\mu\nu}$:

- Time translation: $\xi^\mu = (1, 0, 0, 0)$
- Spatial translation: $\xi^\mu = (0, 1, 0, 0)$



- Rotation about z-axis: $\xi^\mu = (0, -y, x, 0)$

Each satisfies the Killing equation. Minkowski spacetime has **10 independent Killing vectors**, corresponding to the Poincaré group.

Let u^μ be the 4-velocity of a freely falling particle (geodesic):

$$u^\nu \nabla_\nu u^\mu = 0$$

If ξ^μ is a Killing vector, then

$$\frac{d}{d\tau}\left(\xi_\mu u^\mu\right) = 0 \tag{26}$$

Proof.

$$\begin{aligned}
\frac{d}{d\tau}(\xi_\mu u^\mu) &= u^\nu \nabla_\nu (\xi_\mu u^\mu) \\
&= u^\nu u^\mu \nabla_\nu \xi_\mu + \xi_\mu u^\nu \nabla_\nu u^\mu \\
&= \frac{1}{2} u^\nu u^\mu (\nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu) \\
&= 0
\end{aligned}$$

8.7.11 Locally Measured Escape Speed in General Relativity

Consider a static, spherically symmetric spacetime with metric

$$ds^2 = -g_{tt}(r)\, dt^2 + g_{rr}(r)\, dr^2 + r^2 d\Omega^2$$

where $g_{tt}(r) > 0$. As the metric is time-independent, it admits a timelike Killing vector

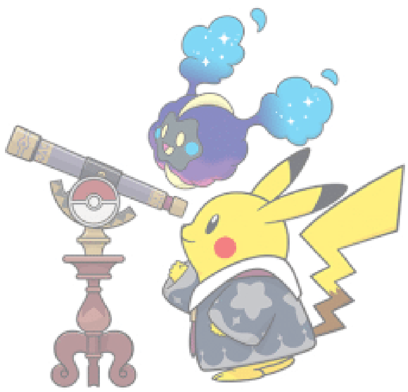
$$K^\mu = (\partial_t)^\mu = (1, 0, 0, 0)$$

Let a particle move along a geodesic with tangent vector

$$u^\mu = \frac{dx^\mu}{d\lambda}$$

If K^μ is a Killing vector, then the quantity

$$K_\mu u^\mu$$



is conserved along the geodesic. We define the conserved energy per unit mass

$$E \equiv -K_\mu u^\mu$$

Since $K^\mu = (1, 0, 0, 0)$, we have

$$K_\mu = g_{\mu\nu} K^\nu = (g_{tt}, 0, 0, 0)$$

and therefore

$$E = -g_{tt} u^t$$

For a massive particle, the four-velocity satisfies

$$u^\mu u_\mu = -1$$

For purely radial motion ($u^\theta = u^\phi = 0$), this gives

$$-g_{tt}(u^t)^2 + g_{rr}(u^r)^2 = -1$$

Substituting $u^t = E/g_{tt}$, we obtain

$$-g_{tt}\left(\frac{E}{g_{tt}}\right)^2 + g_{rr}(u^r)^2 = -1$$

which simplifies to

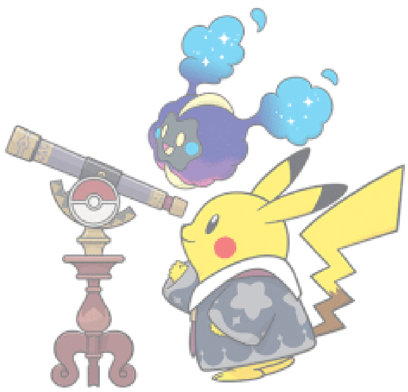
$$g_{rr}(u^r)^2 = \frac{E^2}{g_{tt}} - 1$$

A static observer at radius r has four-velocity

$$U^\mu = \frac{1}{\sqrt{g_{tt}}}(1, 0, 0, 0)$$

The locally measured speed v of the particle is defined by

$$v^2 = \frac{(\text{proper spatial distance})^2}{(\text{proper time})^2} = \frac{g_{rr}(u^r)^2}{g_{tt}(u^t)^2}$$



Using the previous results, this becomes

$$v^2 = \frac{\frac{E^2}{g_{tt}} - 1}{\frac{E^2}{g_{tt}}} = 1 - \frac{g_{tt}}{E^2}$$

For a particle that just escapes to infinity in an asymptotically flat spacetime,

$$g_{tt}(\infty) = 1 \qquad u^r(\infty) = 0$$

The normalization condition then implies

$$E = 1$$

Substituting into the expression for v^2, we obtain the local escape speed:

$$v_{\text{esc}}^2 = 1 - g_{tt}(r)$$

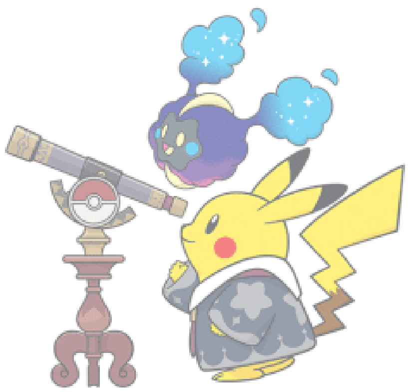
For the Schwarzschild metric,

$$g_{tt}(r) = 1 - \frac{2GM}{r}$$

Hence,

$$v_{\text{esc}}^2 = \frac{2GM}{r}$$

This is the locally measured escape speed by a static observer using proper rulers and clocks.



9 Thermodynamics

9.1 Kelvin

Definition. 9.1: Kelvin

The **Kelvin (K)** is the SI unit of thermodynamic temperature such that

$$T_\mathrm{K} = T_\mathrm{C} + 273.15 \tag{27}$$

where T_C refers to the Celsius scale.

Kelvin must be used when formulas involve energy, ratios, or exponential dependence on temperature.

9.2 Pressure and Hydrostatic Equilibrium

Introduction Pressure is a scalar quantity defined as the force applied per unit area. Mathematically,

$$P = \frac{F}{A}$$

Hydrostatic Equilibrium: Plane-Parallel Case Consider a small cylindrical fluid element of cross-sectional area A and height dz.

- **Gravitational force (downward):** The mass of the element is $m = \rho A\, dz$. Hence,

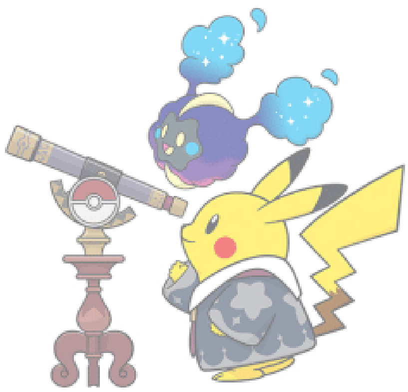
$$F_g = mg = \rho g A\, dz$$

- **Pressure force (upward):** The pressure at the bottom is $P(z)$, while at the top it is $P(z + dz) = P + dP$. The net pressure force is therefore

$$F_{P,\mathrm{net}} = P(z)A - P(z + dz)A = -\, AdP$$

For hydrostatic equilibrium, the total force must vanish:

$$\sum F = 0 \quad \Longrightarrow \quad F_{P,\mathrm{net}} - F_g = 0$$



Hence,

$$-A\,dP - \rho g A\,dz = 0 \quad \Longrightarrow \quad \boxed{\frac{dP}{dz} = -\,\rho g}$$

Hydrostatic Equilibrium: Spherical Symmetry (Stars) Now consider a thin spherical shell inside a star at radius r with thickness dr.

- **Gravitational force (inward):** The mass of the shell is

$$dm = \rho(r)\,dV = \rho(r)\,4\pi r^2 dr$$

The gravitational acceleration at radius r is

$$g(r) = \frac{GM(r)}{r^2}$$

where $M(r)$ is the mass enclosed within radius r. Hence, the inward gravitational force is

$$F_g = dm\,g(r) = 4\pi G\rho(r)M(r)\,dr$$

- **Pressure force (outward):** The surface area of the shell is $A = 4\pi r^2$. The net pressure force is the difference between the outward pressure at r and that at $r + dr$:

$$F_{P,\text{net}} = P(r)A - P(r+dr)A = -\,\frac{dP}{dr}\cdot 4\pi r^2 dr$$

Hydrostatic equilibrium requires

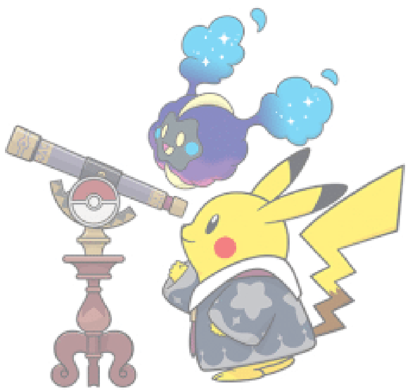
$$\sum F = 0 \quad \Longrightarrow \quad F_{P,\text{net}} - F_g = 0$$

Therefore,

$$-\frac{dP}{dr}\,4\pi r^2 dr - 4\pi G\rho(r)M(r)\,dr = 0$$

which yields the stellar hydrostatic equilibrium equation:

$$\boxed{\frac{dP}{dr} = -\,\frac{GM(r)\rho(r)}{r^2}}$$



Remark. In addition to hydrostatic equilibrium, stars also satisfy **thermal equilibrium**, meaning that the energy generated in the core equals the energy radiated from the surface:

$$L_{\text{core}} = L_{\text{surface}}$$

where L denotes the luminosity. Stellar equilibrium refers to the combined balance of gravitational, pressure, and energy-transport processes that allow a star to maintain a stable structure over long timescales.

9.3 Ideal Gas Law

The Ideal Gas Law is a fundamental equation in thermodynamics that describes the behavior of an ideal gas. It establishes a relationship between the pressure P, volume V, temperature T, and the number of moles of gas n. The equation is expressed as

$$PV = nRT$$

where

- P is the pressure of the gas,
- V is the volume of the gas,
- n is the number of moles of gas,
- R is the universal gas constant ($R = 8.314\,\text{J/mol K}$),
- T is the temperature of the gas in Kelvin.

For a gas undergoing a change in volume at constant pressure, the work done by the gas is given by

$$W = P\Delta V$$

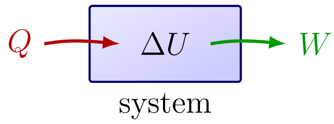
9.4 First Law of Thermodynamics

It states that energy cannot be created or destroyed, only transformed. It is expressed mathematically as

$$dU = \delta Q - \delta W$$

where

- dU is the change in internal energy of the system,
- δQ is the heat added to the system,
- δW is the work done by the system.



Process	**Condition**	**Description**
Adiabatic	$\delta Q = 0$	No heat exchange with surroundings; temperature changes due to work done
Isothermal	$T = \text{constant}$	Temperature remains constant; heat absorbed equals work done
Isobaric	$P = \text{constant}$	Pressure remains constant; volume changes with temperature
Isochoric (Isovolumetric)	$V = \text{constant}$	No work done since volume is constant; heat changes internal energy
Polytropic	$PV^n = \text{constant}$	Generalized process covering isothermal, adiabatic, and isobaric cases
Cyclic	Final state = Initial state	Net change in internal energy over one cycle is zero
Reversible	Quasi-static with no losses	Idealized process that delivers maximum possible work
Irreversible	Finite gradients and friction	Real processes; entropy production is positive

Table 2: Thermodynamic processes, their conditions, and descriptions

9.5 Second Law of Thermodynamics

Definition. 9.2: Entropy

Entropy is a measure of the disorder or randomness of a system. For a reversible process, the change in entropy dS is given by the heat transferred dQ divided by the temperature T:

$$dS = \frac{dQ_{\text{rev}}}{T}$$

Theorem. 9.1: Sackur–Tetrode Equation

$$S = Nk_B \left(\ln \left(\frac{V}{N} \left(\frac{4\pi mE}{3h^2} \right)^{3/2} \right) + \frac{5}{2} \right)$$

where

- S is the entropy,
- N is the number of particles,
- V is the volume,
- m is the mass of a gas particle, and
- E is the internal energy

Theorem. 9.2: Second Law of Thermodynamics

The second law of thermodynamics states that the entropy of an isolated system tends to increase over time.

$$dS \geq 0$$

Theorem. 9.3: Gibbs Free Energy

The Gibbs free energy G is related to entropy through the following equation:

$$G = H - TS = U + PV - TS$$

where H is the enthalpy, T is the temperature (kept constant), P is the pressure (kept constant), and S is the entropy.

9.6 Heat Capacity

Definition. 9.3: Heat Capacity

The heat capacity at constant volume, C_V, is defined as the amount of heat required to raise the temperature of a system by one degree Celsius (or one Kelvin) while maintaining constant volume. Mathematically, it is expressed as

$$C_V = \left(\frac{\partial Q}{\partial T}\right)_V$$

where Q is the heat added to the system, and T is the temperature. The subscript V indicates that the volume is held constant.

Since the system is not allowed to do any work (because the volume is fixed), the heat added to the system only changes the internal energy:

$$dQ = dU$$

where dU is the change in internal energy.
For an ideal gas, the heat capacity at constant volume is related to the molar heat capacity C_V by

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V$$

For an ideal gas, the internal energy U is a function of temperature alone, and the specific heat capacity can be derived from the equation of state.

Definition. 9.4: Heat Capacity

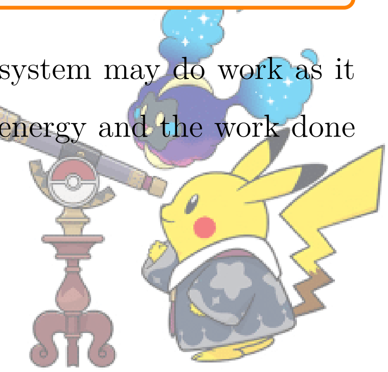
The heat capacity at constant pressure, C_P, is defined as the amount of heat required to raise the temperature of a system by one degree Celsius (or one Kelvin) while maintaining constant pressure. Mathematically, it is expressed as

$$C_P = \left(\frac{\partial Q}{\partial T}\right)_P$$

where Q is the heat added to the system, and T is the temperature. The subscript P indicates that the pressure is held constant.

In contrast to constant volume, when heat is added at constant pressure, the system may do work as it expands. Therefore, the total heat added is the sum of the change in internal energy and the work done by the system

$$dQ = dU + PdV$$



For an ideal gas, the heat capacity at constant pressure is related to the molar heat capacity C_P by

$$C_P = \left(\frac{\partial H}{\partial T}\right)_P$$

Theorem. 9.4: Mayer's Relation

$$c_P - c_V = nR$$

where n is the number of moles of the gas and R is the universal gas constant.

Proof. By the first law of thermodynamics,

$$dU = \delta Q - pdV = nC_V dT$$

where C_V is the heat capacity at constant volume, and n is the number of moles. For the heat added at constant pressure, we use:

$$\delta Q = nC_P dT$$

where C_P is the heat capacity at constant pressure.
By the ideal gas equation of state,

$$pV = nRT$$

Differentiating this equation with respect to temperature at constant pressure:

$$pdV + Vdp = nRdT$$

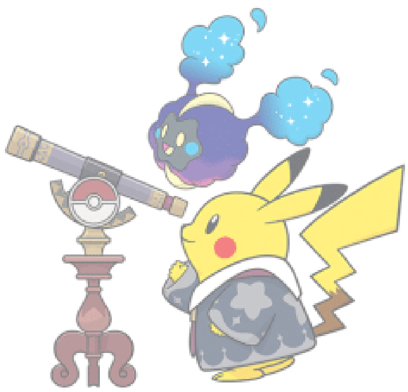
At constant pressure, $dp = 0$, so we have:

$$pdV = nRdT$$

Substitute $pdV = nRdT$ into the equation:

$$nC_V dT = nC_P dT - nRdT$$

$$C_V = C_P - R \implies \boxed{C_P - C_V = nR}$$



Definition. 9.5: Heat Capacity Ratio

$$\gamma = \frac{C_p}{C_V}$$

In an adiabatic process, there is no heat exchange with the surroundings ($dQ = 0$). The first law of thermodynamics gives:

$$dU = -PdV$$

For an ideal gas, the relationship between pressure and volume during an adiabatic process is governed by the equation:

$$PV^\gamma = \text{constant}$$

9.7 Blackhole Thermodynamics

Black hole entropy is a measure of the amount of information that is hidden inside a black hole. The famous Bekenstein-Hawking entropy formula links the entropy of a black hole to the area of its event horizon, rather than its volume. This result was first derived by Jacob Bekenstein and later confirmed by Stephen Hawking:

$$S = \frac{k_B c^3 A}{4G\hbar}$$

Proof. The first law of black hole thermodynamics is given by

$$dM = TdS + \Phi dQ + \Omega dJ$$

where M is the mass, T is the temperature, S is the entropy, Φ is the electrostatic potential, Q is the charge, and J is the angular momentum.
For a non-rotating, uncharged Schwarzschild black hole, the first law reduces to

$$dM = TdS$$

which is similar to the first law of thermodynamics, and implies that the black hole mass is related to its entropy via its temperature. The area of the event horizon is

$$A = 16\pi \left(\frac{GM}{c^2}\right)^2$$

Using the fact that the temperature T of a Schwarzschild black hole is given by the Hawking temperature,

$$T = \frac{\hbar c^3}{8\pi GM} \text{ (The derivation is out of scope of this note.)}$$

The first law $dM = TdS$ implies

$$dS = \frac{dM}{T} = \frac{dM}{\frac{\hbar c^3}{8\pi GM}} = \frac{8\pi GM}{\hbar c^3} dM$$

Integrating with respect to M,

$$S = \frac{8\pi GM^2}{\hbar c^3}$$

Using the expression for the area $A = 16\pi \left(\frac{GM}{c^2}\right)^2$,

$$M = \frac{c^2\sqrt{A}}{4\pi G}$$

Substituting this into the entropy expression, we get

$$S = \frac{k_B c^3 A}{4G\hbar}$$

9.8 Kinetic Theory

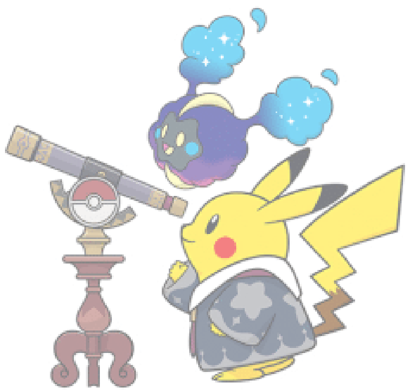
9.8.1 Mean Free Path

Let the number density of the gas (the number of particles per unit volume) be n, and let the cross-sectional area for a collision between two particles be σ. The cross-sectional area σ depends on the type of collision and the physical properties of the particles involved. If we assume spherical particles with radius r, the collision cross-section is given by

$$\sigma = \pi(2r)^2 = 4\pi r^2$$

Next, the relative velocity between two particles in the gas is on the order of the mean speed of the particles v_{avg}. The total rate of collisions per unit time per particle is

$$\text{Collision rate} = n\sigma v_{\text{avg}}$$



The mean free path λ is defined as the average distance a particle travels before undergoing a collision. The relationship between the mean free path and the collision rate is given by

$$\lambda = \frac{1}{\text{Collision rate}} = \frac{1}{n\sigma v_{\text{avg}}} = \frac{1}{n \cdot 4\pi r^2 \cdot v_{\text{avg}}}$$

9.8.2 Boltzmann Distribution and Maxwell–Boltzmann Distribution

The Boltzmann distribution, also known as the Gibbs distribution, is a fundamental probability distribution in statistical mechanics that describes the statistical properties of a system in thermal equilibrium at a fixed temperature. It provides the probability that a system will be in a particular microstate with energy E_i when it is in contact with a heat bath at temperature T.

Formulation and Derivation Consider a system in thermal equilibrium with a large reservoir at constant temperature T. The probability P_i of finding the system in a particular microstate i with energy E_i is given by

$$P_i = \frac{1}{Z} e^{-\beta E_i}$$

where

- $\beta = \dfrac{1}{k_B T}$ is the thermodynamic beta
- $Z = \sum_i e^{-\beta E_i}$ is the partition function

The partition function Z serves as a normalization constant ensuring that $\sum_i P_i = 1$. It contains all thermodynamic information about the system.

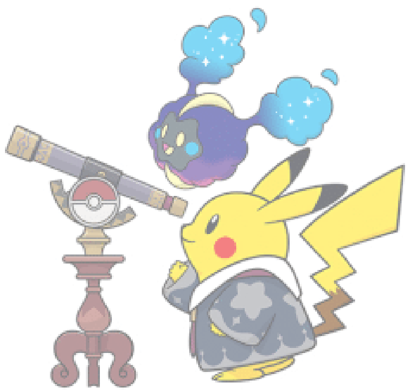
Proof. Statistical mechanics often uses the principle of maximum entropy. The entropy of a discrete set of probabilities $\{P_i\}$ is

$$S = -k_B \sum_i P_i \ln P_i$$

as $S = k_B \ln \Omega$ (where Ω is the number of micro-states by Statistical Mechanics). We want to maximize S subject to the constraints:

$$\sum_i P_i = 1 \quad \text{(normalization)}$$

$$\sum_i P_i E_i = \langle E \rangle \quad \text{(average energy fixed)}$$



Using Lagrange multipliers α and β, define

$$\mathcal{L} = -k_B \sum_i P_i \ln P_i - \alpha \left(\sum_i P_i - 1 \right) - \beta \left(\sum_i P_i E_i - \langle E \rangle \right)$$

Setting the derivative with respect to P_i to zero:

$$\frac{\partial \mathcal{L}}{\partial P_i} = -k_B(\ln P_i + 1) - \alpha - \beta E_i = 0$$

which gives

$$P_i = e^{-1-\alpha/k_B} e^{-\beta E_i/k_B}$$

Defining $Z = e^{1+\alpha/k_B} = \sum_i e^{-\beta E_i/k_B}$, we get the familiar form:

$$P_i = \frac{e^{-\beta E_i}}{Z}$$

Maxwell-Boltzmann Distribution The Maxwell–Boltzmann distribution describes the statistical distribution of speeds (or velocities) of particles in an ideal gas at thermal equilibrium. It is a special case of the Boltzmann distribution applied to the translational kinetic energy of non-interacting particles.
For an ideal gas of N identical particles of mass m at temperature T, the probability density function for finding a particle with velocity $\mathbf{v} = (v_x, v_y, v_z)$ is

$$f(\mathbf{v}) = \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp\left(-\frac{m|\mathbf{v}|^2}{2k_B T} \right)$$

This three-dimensional distribution factorizes as $f(\mathbf{v}) = f(v_x)f(v_y)f(v_z)$, where each component distribution is Gaussian

$$f(v_\alpha) = \sqrt{\frac{m}{2\pi k_B T}} \exp\left(-\frac{m v_\alpha^2}{2k_B T} \right), \quad \alpha = x, y, z$$

More commonly used is the distribution of speeds $v = |\mathbf{v}|$, obtained by integrating over all directions in velocity space:

$$f(v) = 4\pi v^2 \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp\left(-\frac{m v^2}{2k_B T} \right)$$

The factor $4\pi v^2$ arises from the spherical shell volume element in velocity space.

Proof. Consider an ideal gas with N particles in thermal equilibrium at temperature T. Each particle has a mass m and a velocity vector $\mathbf{v} = (v_x, v_y, v_z)$. The total energy of a particle is purely kinetic:

$$E = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2)$$

According to the Boltzmann distribution, the probability of a particle having energy E is

$$P(E) \propto e^{-E/k_B T}$$

Note that

$$P(v_x, v_y, v_z) = f(v_x)f(v_y)f(v_z)$$

$$f(v_i) = Ae^{-\frac{1}{2}mv_i^2/k_B T}, \quad i = x, y, z$$

As

$$\int_{-\infty}^{\infty} f(v_x)\, dv_x = 1, \quad \int_{-\infty}^{\infty} Ae^{-\frac{mv_x^2}{2k_B T}} dv_x = 1$$

and by

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

we have

$$A\sqrt{\frac{2\pi k_B T}{m}} = 1 \quad \Longrightarrow \quad A = \sqrt{\frac{m}{2\pi k_B T}}$$

Hence,

$$\boxed{f(v_i) = \sqrt{\frac{m}{2\pi k_B T}} e^{-\frac{mv_i^2}{2k_B T}}}$$

The speed of a particle is

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

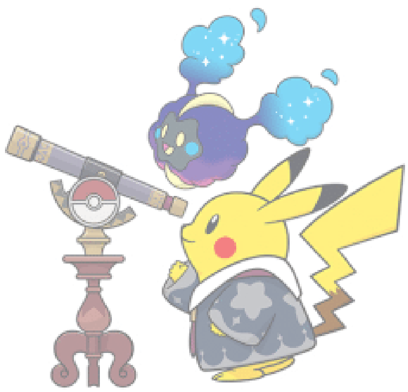
The probability density function of speeds, $F(v)$, can be obtained by transforming to spherical coordinates in velocity space:

$$F(v)dv = 4\pi v^2 f(v_x)f(v_y)f(v_z)dv$$

Hence,

$$\boxed{F(v) = 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 e^{-\frac{mv^2}{2k_B T}}}$$

Three important characteristic speeds are derived from this distribution:



1. **Most probable speed** (mode):

$$v_{\text{mp}} = \sqrt{\frac{2k_B T}{m}} = \sqrt{\frac{2RT}{M}}$$

2. **Mean speed**:

$$\langle v \rangle = \sqrt{\frac{8k_B T}{\pi m}} = \sqrt{\frac{8RT}{\pi M}}$$

3. **Root-mean-square speed**:

$$v_{\text{rms}} = \sqrt{\frac{3k_B T}{m}} = \sqrt{\frac{3RT}{M}}$$

where $R = N_A k_B$ is the gas constant and $M = N_A m$ is the molar mass.

9.9 Boundary Conditions

Solving the stellar structure equations requires proper boundary conditions:

- **At the Center** ($r = 0$):

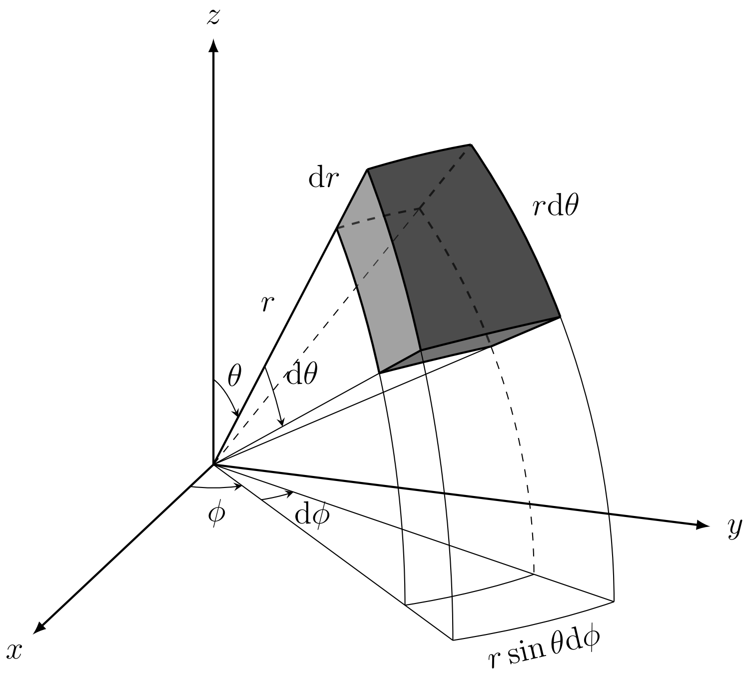
$$M(0) = 0, \quad L(0) = 0$$

- **At the Surface** ($r = R$):

$$P(R) = P_{\text{surface}} \approx 0, \quad T(R) = T_{\text{eff}}$$

These conditions ensure the solution matches physical reality: zero mass at the center, and finite temperature and pressure at the surface.

9.10 Case Study



Suppose a static spherical star consists of N neutral particles with radius R with $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, satisfying the following equation of states

$$PV = Nk\frac{T_R - T_0}{\ln(T_R/T_0)} \tag{1}$$

where P and V are the pressure inside the star and the volume of the star respectively, k is the Boltzmann constant, T_R and T_0 are the temperatures at the surface $r = R$ and the temperature at the center $r = 0$ respectively. Assume that $T_R \leq T_0$.

(a) Simplify the stellar equation of state (1) if $\Delta T = T_R - T_0 \to 0$ (this is called ideal star) (Hint: Use the approximation $\ln(1 + x) \approx x$ for small x).

Suppose the star undergoes a quasi-static process, in which it may slightly contract or expand, such that the above stellar equation of state (1) still holds.

The star satisfies the first law of thermodynamics:

$$Q = \Delta Mc^2 + W \tag{2}$$

where Q, M, and W are heat, mass of the star, and work respectively, while c is the light speed in the vacuum and $\Delta M = M_{\text{final}} - M_{\text{initial}}$.

In the following, we assume T_0 to be constant, while $T_R \equiv T$ varies.

(b) Find the heat capacity of the star at constant volume C_V and at constant pressure C_P, expressed in C_P and C_V (Hint: Use the approximation $(1+x)^n \approx 1 + nx$ for small x).

Assuming that C_P is constant and the gas undergoes the isobaric process so the star produces the heat and radiates it outside to the space.

(c) Find the heat produced by the isobaric process if the initial temperature and the final temperature are T_i and T_f, respectively.

(d) For the next parts, assume the star is the Sun.

(e) If the sunlight is monochromatic with frequency 5×10^{14} Hz, estimate the number of photons radiated by the Sun per second.

(f) Calculate the heat capacity C_P of the Sun assuming its surface temperature varies from 5500 K to 6000 K in one second.

Solution.

(a) Defining $\Delta T = T_\mathrm{f} - T_0$ and $\Delta T \approx 0$, we have

$$\frac{P\Delta V}{Nk} = \frac{\Delta T}{\ln(1 + \Delta T/T_0)}$$

Using $\ln(1 + \Delta T/T_0) \approx \Delta T/T_0$, we then obtain

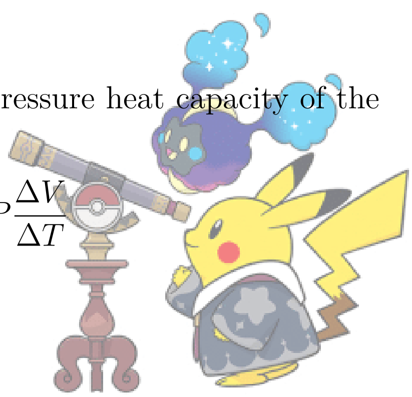
$$\frac{P\Delta V}{Nk} = T_0$$

(b) The internal energy of the star is $U = Mc^2 (U(T) = M(T)c^2$ for ideal star). Hence, the constant volume heat capacity of the star has the form:

$$C_V = \left(\frac{\Delta Q}{\Delta T}\right)_V = \left(\frac{\Delta M}{\Delta T}\right)_V c^2$$

for small ΔT. Then, using first law of thermodynamics, the constant pressure heat capacity of the star is

$$C_P = \left(\frac{\Delta Q}{\Delta T}\right)_P = \left(\frac{\Delta M}{\Delta T}\right)_V c^2 + P\frac{\Delta V}{\Delta T} = C_V + P\frac{\Delta V}{\Delta T}$$



for small ΔT. Defining $\Delta T = T_2 - T_1$, then

$$\frac{P\Delta V}{Nk} = \frac{T_1 - T_0 + \Delta T}{\ln((T_1 + \Delta T)/T_0)} - \frac{T_1 - T_0}{\ln(T_1/T_0)}$$

Using the approximation

$$\ln((T_1 + \Delta T)/T_0) \approx \ln\left(\frac{T_1}{T_0}\right) + \frac{\Delta T}{T_1}$$

$$\frac{1}{\ln((T_1 + \Delta T)/T_0)} \approx \frac{1}{\ln(T_1/T_0)} - \frac{1}{T_1 \ln(T_1/T_0)^2} \frac{\Delta T}{T_1}$$

then we have

$$\frac{P\Delta V}{\Delta T} \approx Nk\left(1 - \frac{1}{\ln(T/T_0)}\right)\frac{(T - T_0)/T}{\ln(T/T_0)}$$

where $T_1 = T$. Finally, we obtain

$$C_P = \left(\frac{\Delta Q}{\Delta T}\right)_P = \left(\frac{\Delta M}{\Delta T}\right)_V c^2 + P\frac{\Delta V}{\Delta T} = C_V + \frac{Nk}{\ln(T/T_0)}\left(1 - \frac{(T - T_0)/T}{\ln(T/T_0)}\right)$$

(c) Since C_V is constant, the heat produced by the star is given by

$$Q_H = C_V(T_\mathrm{f} - T_\mathrm{i}) + P\Delta V$$

$$Q_H = C_V(T_\mathrm{f} - T_\mathrm{i}) + Nk\left[\frac{T_\mathrm{f} - T_0}{\ln(T_\mathrm{f}/T_0)} - \frac{T_\mathrm{i} - T_0}{\ln(T_\mathrm{i}/T_0)}\right]$$

(d) Energy per second radiated by the Sun $\dot{E} = L_\odot = Nh\nu$ where N is the number of photons. Hence

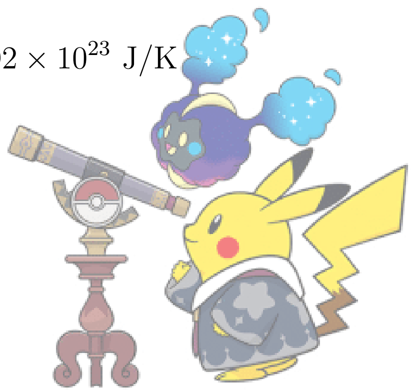
$$N = \frac{L_\odot}{h\nu} = \frac{3.90 \times 10^{26}}{6.626 \times 10^{-34} \times 5 \times 10^{14}} = 1.195 \times 10^{45} \text{ photons}$$

(e) Energy per second radiated by the Sun is proportional to mass defect of the Sun

$$L_\odot = \frac{\Delta M c^2}{\Delta t}$$

Hence,

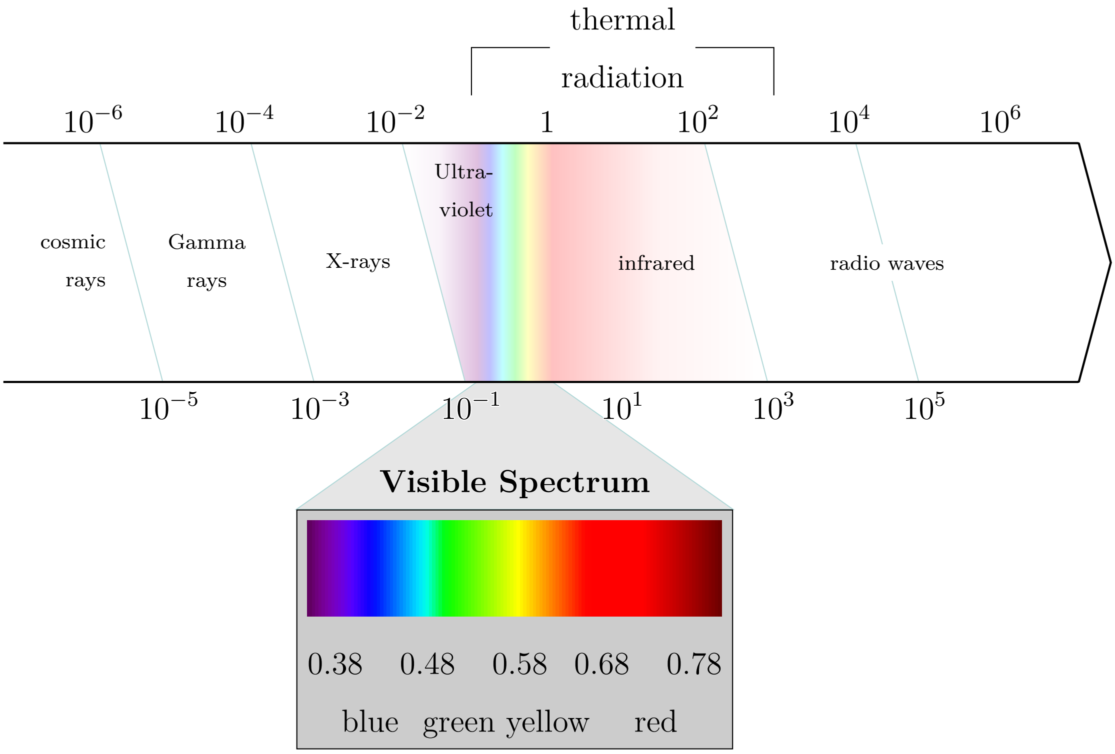
$$C_V = \frac{\Delta M c^2}{L_\odot} = \frac{3.96 \times 10^{26}}{\Delta T}\frac{1}{\frac{L_\odot}{\Delta t}} \approx \frac{3.96 \times 10^{26}}{6000 - 5500} \text{ J/K} = 7.92 \times 10^{23} \text{ J/K}$$



10 Spectroscopy

10.1 Basic Concepts

- **(Spectrum)** The diffraction of light produces a spectrum, which can be observed as a series of bright and dark fringes.



- **(Absorption)** Absorption occurs when atoms or molecules in a celestial object absorb photons of specific energies, raising electrons to higher energy levels.

- **(Emission)** Emission occurs when excited electrons drop to lower energy levels, releasing photons of specific energies, forming **emission lines**.

$$E_{\text{photon}} = E_{\text{upper}} - E_{\text{lower}}$$

This leads to **absorption lines** in the spectrum.

(4) Spectral lines reveal chemical composition, temperature, pressure, and velocity fields (via Doppler shifts).

(5) **(Scattering)** Scattering occurs when photons interact with particles, changing direction and sometimes energy:

- **Rayleigh scattering:** Elastic scattering by particles much smaller than wavelength.
- **Thomson scattering:** Elastic scattering by free electrons.
- **Compton scattering:** Inelastic scattering, photon loses energy. There is a relation derived by Compton:

$$\boxed{\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta)}$$

where

(i) λ is the initial wavelength of the photon,

(ii) λ' is the wavelength of the scattered photon,

(iii) h is Planck's constant,

(iv) m_e is the mass of the electron,

(v) c is the speed of light,

(vi) θ is the scattering angle, which is the angle between the direction of the incident and scattered photon.

(6) **(Splitting)** Splitting occurs when a single spectral line divides into multiple components due to external or internal interactions. The Zeeman effect is the splitting of spectral lines in the presence of a magnetic field **B**. For transitions with no spin, a single spectral line splits into three components:

$$\Delta E = m_l \mu_B B, \quad m_l = 0, \pm 1$$

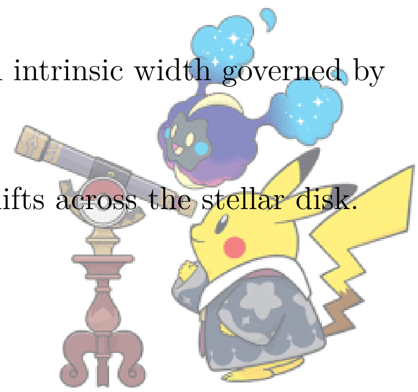
where μ_B is the Bohr magneton, B is the magnetic field strength, and m_l is the magnetic quantum number.

The Stark effect is the splitting of spectral lines due to an external electric field **E**:

$$\Delta E \propto E$$

(7) **(Broadening)** Broadening refers to the widening of spectral lines beyond their natural linewidth.

- Due to the finite lifetime τ of excited states, spectral lines have an intrinsic width governed by the uncertainty principle.
- Stellar rotation causes line broadening due to different Doppler shifts across the stellar disk.



10.2 Spectral Radiance

The spectral radiance (power emitted per unit area per unit solid angle per unit wavelength) is given by Planck's law:

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/(\lambda k_B T)} - 1}$$

where λ is the wavelength.

Astronomy has evolved from visible-light observations to encompass the entire electromagnetic (EM) spectrum. Each wavelength band reveals unique astrophysical phenomena, as expressed by Planck's law.

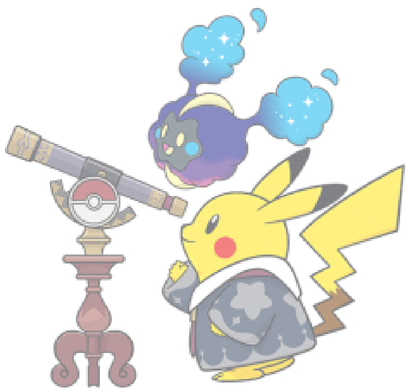
Band	**Wavelength Range**	**Energy Range**	**Temperature (K)**	**Primary Sources**
Radio	$> 10\,\text{cm}$	$< 1.24 \times 10^{-5}\,\text{eV}$	< 0.1	Cold gas, pulsars
Microwave	$1\,\text{mm} - 10\,\text{cm}$	$1.24 \times 10^{-5}\,\text{eV} - 1.24 \times 10^{-3}\,\text{eV}$	$0.1 - 10$	CMB, molecular clouds
Infrared	$700\,\text{nm} - 1\,\text{mm}$	$1.24 \times 10^{-3}\,\text{eV} - 1.77\,\text{eV}$	$10 - 10^4$	Dust, planets, cool stars
Visible	$400\,\text{nm} - 700\,\text{nm}$	$1.77\,\text{eV} - 3.1\,\text{eV}$	10^4	Stars, galaxies, nebulae
Ultraviolet	$10\,\text{nm} - 400\,\text{nm}$	$3.1\,\text{eV} - 124\,\text{eV}$	$10^4 - 10^5$	Hot stars, quasars
X-ray	$0.01\,\text{nm} - 10\,\text{nm}$	$124\,\text{eV} - 124\,\text{keV}$	$10^5 - 10^8$	Black holes, supernovae
Gamma-ray	$< 0.01\,\text{nm}$	$> 124\,\text{keV}$	$> 10^8$	GRBs, nuclear processes

Figure 8: Electromagnetic spectrum bands in astronomy

Limiting cases:

- **Rayleigh-Jeans Law (long wavelengths, $hc \ll \lambda k_B T$):**

$$B_\lambda(T) \approx \frac{2ck_B T}{\lambda^4}$$



- **Wien's Law (short wavelengths, $hc \gg \lambda k_B T$):**

$$B_\lambda(T) \approx \frac{2hc^2}{\lambda^5} e^{-hc/(\lambda k_B T)}$$

The wavelength at which the emission is maximum:

$$\lambda_{\text{max}} T \approx 2.898 \times 10^{-3} \text{m K} \textbf{ (Wien's Displacement Law)}$$

10.3 Stefan–Boltzmann law

$$j^* = \sigma T^4$$

where

- j^* is the total radiative flux (power per unit area, (W/m^2)),
- T is the absolute temperature of the blackbody in Kelvin (K),
- σ is the Stefan–Boltzmann constant:

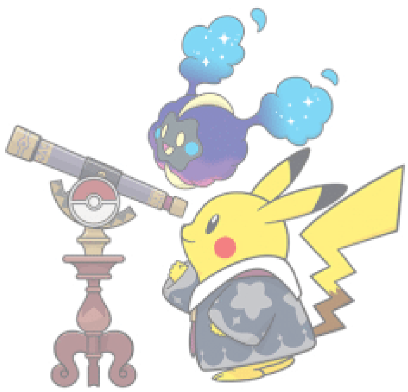
$$\sigma = 5.670374419 \times 10^{-8} \text{ Wm}^{-2}\text{K}^{-4}$$

For a real object with emissivity (ϵ) $(0 \leq \epsilon \leq 1)$, the law generalizes to

$$j = \epsilon \sigma T^4$$

Proof. The total radiated power per unit area is obtained by integrating over all frequencies and solid angles from Planck's law:

$$\begin{aligned} j^* &= \int_0^\infty \int_\Omega B(\nu, T) \cos\theta d\Omega d\nu \\ &= \pi \int_0^\infty B(\nu, T) d\nu \end{aligned}$$



Let $x = \dfrac{h\nu}{k_B T}$.

$$
\begin{aligned}
j^* &= \frac{2\pi h}{c^2} \int_0^\infty \frac{\nu^3}{e^{h\nu/k_B T} - 1} d\nu \\
&= \frac{2\pi (k_B T)^4}{h^3 c^2} \int_0^\infty \frac{x^3}{e^x - 1} dx
\end{aligned}
$$

The integral evaluates to $\dfrac{\pi^4}{15}$, giving

$$
j^* = \sigma T^4, \quad \sigma = \frac{2\pi^5 k_B^4}{15 h^3 c^2}
$$

10.4 Doppler's Effect

The **Doppler effect** is a fundamental phenomenon in wave physics where the observed frequency of a wave changes due to relative motion between the source and observer.

For non-relativistic speeds ($v \ll c$), where c is the speed of light:

$$
f_{\text{obs}} = \frac{f_{\text{src}}}{1 \pm \frac{v_s}{c}} \text{ (Source Moving, Observer Stationary)}
$$

where $+$ for receding source, $-$ for approaching source.

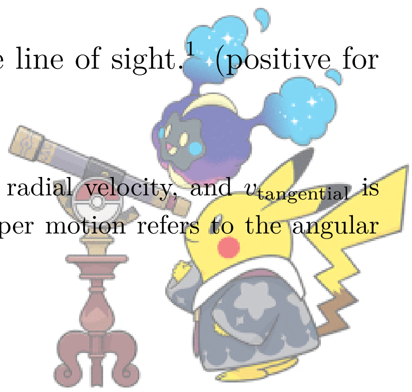
$$
f_{\text{obs}} = f_{\text{src}} \left(1 \pm \frac{v_o}{c}\right) \text{ (Observer Moving, Source Stationary)}
$$

In astrophysics, we typically measure the **radial velocity** v_r through the wavelength shift:

$$
\frac{\Delta\lambda}{\lambda_0} = \frac{\lambda_{\text{obs}} - \lambda_0}{\lambda_0} = \frac{v_r}{c}
$$

where

- λ_0 represents the rest wavelength,
- λ_{obs} represents the observed wavelength, and
- v_r represents the radial velocity which is the motion of a star along the line of sight.[1] (positive for recession)



[1] The total velocity of a star is given by $v = \sqrt{v_{\text{radial}}^2 + v_{\text{tangential}}^2}$ where v_{radial} is the radial velocity, and $v_{\text{tangential}}$ is the tangential velocity, which can be derived from proper motion and distance. Proper motion refers to the angular movement of a star across the sky, usually measured in arcseconds per year.

For astronomical objects moving at significant fractions of light speed, we must use the relativistic formula:

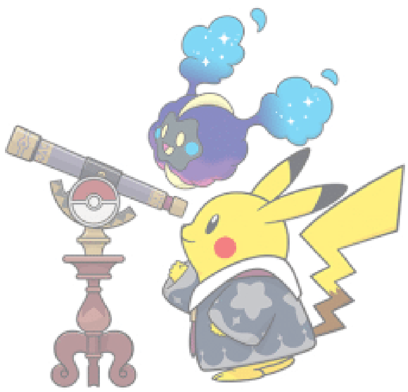
$$\frac{\lambda_{\text{obs}}}{\lambda_0} = \sqrt{\frac{1+\beta}{1-\beta}} \quad \text{or} \quad \frac{f_{\text{obs}}}{f_0} = \sqrt{\frac{1-\beta}{1+\beta}}$$

The redshift z is defined as

$$z = \frac{\lambda_{\text{obs}} - \lambda_0}{\lambda_0} = \frac{\Delta\lambda}{\lambda_0}$$

For relativistic speeds,

$$1 + z = \sqrt{\frac{1+\beta}{1-\beta}}$$



11 Electromagnetism

11.1 Lorentz's Force

The Lorentz force law describes the force exerted on a charged particle moving through an electromagnetic field. It is given by

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

- $\mathbf{F}$ is the force on the charged particle,
- q is the charge of the particle,
- $\mathbf{E}$ is the electric field,
- $\mathbf{v}$ is the velocity of the particle, and
- $\mathbf{B}$ is the magnetic field.

11.2 Maxwell's Equation

Definition. 11.1: Del Operator

For continuously differentiable vector field $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$, define $\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$. Then

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \text{ and } \nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k}$$

which $\nabla \cdot \mathbf{F}$ measures the net flow of the field out of an infinitesimally small volume around a point.

Theorem. 11.1: Differential Form of Maxwell's Equation

For free charge density ρ (S.I. unit: $\mathrm{C/m^3}$) and free current density $\mathbf{J}$ (S.I. unit: $\mathrm{A/m^2}$),

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \text{(Gauss's Law for Electricity)}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(Gauss's Law for Magnetism)}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Faraday's Law of Induction)}$$

$$\nabla \times \mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t} \qquad \text{(Ampère-Maxwell Law)}$$

Theorem. 11.2: Integral Form of Maxwell's Equation

For free charge density ρ and free current density $\mathbf{J}$, the Maxwell's equations in integral form are:

$$\int_S \mathbf{E} \cdot d\mathbf{A} = \frac{1}{\epsilon_0} \int_V \rho \, dV \qquad \text{(Gauss's Law for Electricity)}$$

$$\int_S \mathbf{B} \cdot d\mathbf{A} = 0 \qquad \text{(Gauss's Law for Magnetism)}$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{A} \qquad \text{(Faraday's Law of Induction)}$$

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{A} + \mu_0 \epsilon_0 \frac{d}{dt} \int_S \mathbf{E} \cdot d\mathbf{A} \qquad \text{(Ampère-Maxwell Law)}$$

11.3 Poynting Vector

The Poynting vector describes the directional energy flux (the energy transfer per unit area per unit time) or power flow of an electromagnetic field. It is given by

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$$

where

- $\mathbf{S}$ is the Poynting vector,
- $\mathbf{E}$ is the electric field, and
- $\mathbf{B}$ is the magnetic field.

11.4 Optics

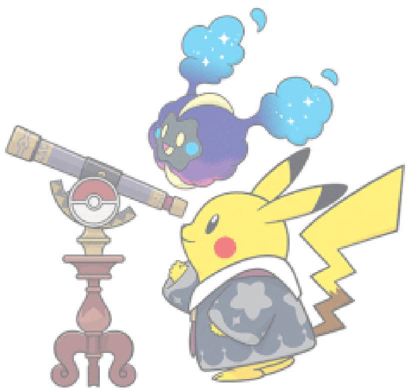
11.4.1 Wavefunction

The displacement of a point on a string in simple harmonic motion can be modeled by a sinusoidal function. The solution to the wave equation for a string under tension is:

$$\phi(x, t) = A \sin(kx - \omega t + \delta)$$

where

- A is the amplitude of the wave,



- $k = \dfrac{2\pi}{\lambda}$ is the wave number (with λ being the wavelength),
- $\omega = 2\pi f$ is the angular frequency (with f being the frequency), and
- δ is a phase constant.

11.4.2 Lens

Theorem. 11.3: Refractive Index

The **refractive index** of a medium is defined as the ratio of the speed of light in vacuum to the phase velocity of light in the medium:

$$n \equiv \frac{c}{v}$$

Without free charge, Ampère's law states that

$$\nabla \times \mathbf{B} = \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

Take the curl of Faraday's law:

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B})$$

Substituting the expression for $\nabla \times \mathbf{B}$ yields

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Using the vector identity

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$$

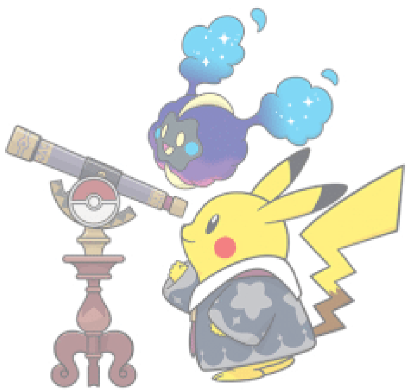
and Gauss's law, we obtain

$$-\nabla^2 \mathbf{E} = -\mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\nabla^2 \mathbf{E} = \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

The standard wave equation for a field $\mathbf{E}$ propagating with speed v is

$$\nabla^2 \mathbf{E} = \frac{1}{v^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}$$



Comparing with the equation above, we identify

$$\frac{1}{v^2} = \mu\varepsilon$$

Therefore, the speed of electromagnetic waves in the medium is

$$v = \frac{1}{\sqrt{\mu\varepsilon}}$$

For speed of light, $c = \dfrac{1}{\sqrt{\epsilon_0\mu_0}}$. Then $n = \sqrt{\epsilon_r\mu_r}$ where $\epsilon_r = \dfrac{\epsilon}{\epsilon_0}$ is the relative permittivity (dielectric constant) of the medium and $\mu_r = \dfrac{\mu}{\mu_0}$ is the relative permeability of the medium. For $\mu_r \approx 1$, $n \approx \sqrt{\epsilon_r}$.

Theorem. 11.4: Snell's Law

Consider an interface between two homogeneous, isotropic media with refractive indices n_1 and n_2. Let θ_1 and θ_2 denote the angles that the incident and refracted rays make with the normal to the interface, respectively. Snell's law states that

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

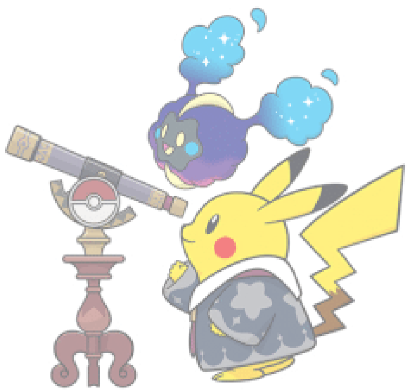
Theorem. 11.5: Lens Formula

$$\frac{1}{f} = \frac{1}{v} + \frac{1}{u}$$

where

- f is the focal length of the lens, which is the distance from the optical element (lens or mirror) to the point where light converges to form an image and determines the magnification and field of view of the telescope,
- u is the object distance (distance from the object to the lens), and
- v is the image distance (distance from the lens to the image).

The most widely used convention is the **Cartesian sign convention**:



Parameter	**Sign Convention**
Object distance (u)	Negative (for real objects)
Image distance (v)	Positive (for real images)
	Negative (for virtual images)
Focal length (f)	Positive (for convex/ converging lenses)
	Negative (for concave/ diverging lenses)
Height of object (h_o)	Positive (upward from principal axis)
Height of image (h_i)	Positive (upward from principal axis)
	Negative (downward from principal axis)

Table 3: Sign conventions for lens formula

Definition. 11.2: Optical Power

For a lens with focal length f, the power is given by

$$P = \frac{1}{f}$$

Theorem. 11.6: Combination of Thin Lens

For two or more thin lenses close together, the effective power is given by $\sum P$.

Theorem. 11.7: Lens Maker's Equation

$$\frac{1}{f} = (n-1)\left[\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n-1)d}{nR_1R_2}\right]$$

where

- f is the focal length of the lens,
- n is the refractive index of the lens material,
- R_1 is the radius of curvature of the first surface of the lens,
- R_2 is the radius of curvature of the second surface of the lens,

- d is the thickness of the lens.

Proof. We will derive the case of thin lens where $d \to 0$.

Consider a **spherical refracting surface** with radius of curvature R separating two media with refractive indices n_1 and n_2. Let C be the **center of curvature** and V be the **vertex** of the spherical surface. We use the sign convention:

- Distances measured from V
- $R > 0$ if C is to the right of V (convex toward object)
- Object distance $u = -s_o$ (negative if object is left of V)
- Image distance $v = s_i$ (positive if image is right of V)

Using the **paraxial (small-angle) approximation**:

$$\sin\theta \approx \theta, \quad \tan\theta \approx \theta$$

for rays making small angles with the optical axis.

Consider a point object O on the optical axis, sending a ray to point A on the spherical surface at height h above the axis. From triangle OAC,

$$\text{Angle of incidence: } \theta_1 = \alpha + \phi$$

From triangle IAC,

$$\text{Angle of refraction: } \theta_2 = \phi - \beta$$

where

$$\alpha \approx \frac{h}{-u}, \quad \beta \approx \frac{h}{v}, \quad \phi \approx \frac{h}{R}$$

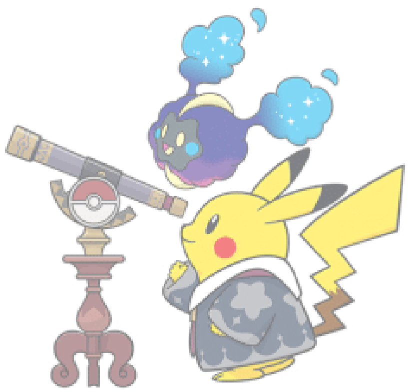
with $u = -s_o < 0$, $v > 0$, and R as given.

Snell's law in paraxial form:

$$n_1\theta_1 = n_2\theta_2$$

Substituting:

$$n_1(\alpha + \phi) = n_2(\phi - \beta)$$



$$n_1\left(\frac{h}{-u}+\frac{h}{R}\right)=n_2\left(\frac{h}{R}-\frac{h}{v}\right)$$

$$n_1\left(\frac{1}{-u}+\frac{1}{R}\right)=n_2\left(\frac{1}{R}-\frac{1}{v}\right)$$

Rewriting with Cartesian sign convention ($u=-s_o$, $v=s_i$, R signed):

$$\boxed{\frac{n_2}{v}-\frac{n_1}{u}=\frac{n_2-n_1}{R}}$$

Consider a **thin lens** of refractive index n_l surrounded by medium n_m. The lens has:

- First spherical surface: radius R_1 (center C_1)
- Second spherical surface: radius R_2 (center C_2)
- Thickness negligible compared to object/image distances (thin lens approximation)

Sign convention:

- Light travels left to right
- $R>0$ if center of curvature is to the right of surface
- Object distance $s_o>0$ (real object left of lens)
- Image distance $s_i>0$ (real image right of lens)

From medium 1 n_m to medium 2 n_l,

$$\frac{n_l}{v_1}-\frac{n_m}{(-s_o)}=\frac{n_l-n_m}{R_1}$$

Since $u_1=-s_o$,

$$\frac{n_l}{v_1}+\frac{n_m}{s_o}=\frac{n_l-n_m}{R_1} \qquad (*)$$

From medium 1 n_l to medium 2 n_m, the object for second surface is the image from first surface. For thin lens, object distance is $-v_1$.

$$\frac{n_m}{s_i}-\frac{n_l}{(-v_1)}=\frac{n_m-n_l}{R_2}$$

$$\frac{n_m}{s_i}+\frac{n_l}{v_1}=\frac{n_m-n_l}{R_2} \qquad (**)$$

From (*), $\frac{n_l}{v_1}=\frac{n_l-n_m}{R_1}-\frac{n_m}{s_o}$. From (**), $\frac{n_l}{v_1}=\frac{n_m-n_l}{R_2}-\frac{n_m}{s_i}$. Equating both expressions for $\frac{n_l}{v_1}$ and simplifying gives

$$\frac{n_m}{s_o}-\frac{n_m}{s_i}=(n_l-n_m)\left(\frac{1}{R_2}-\frac{1}{R_1}\right)$$

For $s_i = f$ (image at focal point),

$$\frac{1}{f} = \frac{n_l - n_m}{n_m}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Let $n_m = 1$ and $n_l = n$.

$$\boxed{\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$$

11.5 Diffraction and Interference

11.5.1 Principle of Superposition

The **principle of superposition** states that when two or more waves overlap in space, the resultant displacement at any point is equal to the algebraic sum of the individual displacements at that point. Let two harmonic waves be given by

$$y_1(x,t) = A_1 \sin(kx - \omega t + \phi_1)$$
$$y_2(x,t) = A_2 \sin(kx - \omega t + \phi_2)$$

According to the principle of superposition, the resultant wave is

$$\begin{aligned} y(x,t) &= y_1(x,t) + y_2(x,t) \\ &= A_1 \sin(kx - \omega t + \phi_1) + A_2 \sin(kx - \omega t + \phi_2) \end{aligned}$$

Using the trigonometric identity

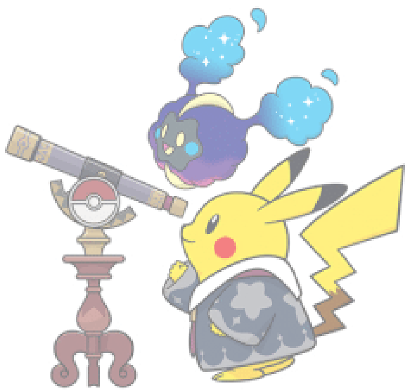
$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

we get

$$y(x,t) = 2A\cos\left(\frac{\phi_2 - \phi_1}{2}\right)\sin\left(kx - \omega t + \frac{\phi_1 + \phi_2}{2}\right)$$

where A is the effective amplitude.

- Constructive interference occurs when $\phi_2 - \phi_1 = 2n\pi$.
- Destructive interference occurs when $\phi_2 - \phi_1 = (2n+1)\pi$.



11.5.2 Complex Number

Recall that a complex number z is written as

$$z = x + iy$$

where x is the real part, y is the imaginary part, and $i = \sqrt{-1}$.

Theorem. 11.8: Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This is extremely useful in wave physics because oscillating quantities like $A\cos(\omega t + \phi)$ can be represented as the real part of a complex exponential:

$$A\cos(\omega t + \phi) = \mathfrak{R}(Ae^{i(\omega t + \phi)})$$

11.5.3 Young's Double-Slit Experiment

Consider two slits separated by distance d illuminated by coherent light of wavelength λ. At a point on a screen at distance L, the path difference (which is defined as the difference in the distance traveled by two waves from their respective sources to a common point.) is

$$\delta = d\sin\theta$$

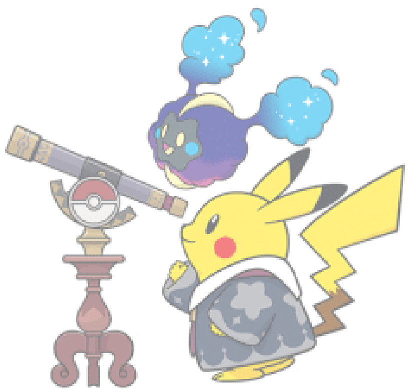
where θ is the angle from the central maximum.

The condition for constructive interference (bright fringes) is

$$d\sin\theta = m\lambda, \quad m = 0, \pm1, \pm2, \ldots$$

For destructive interference (dark fringes),

$$d\sin\theta = \left(m + \frac{1}{2}\right)\lambda$$



For small angles, $\sin\theta \approx \tan\theta = \dfrac{x}{L}$, where x is the fringe displacement. Then the fringe width is

$$\Delta x = \frac{\lambda L}{d}$$

11.5.4 Single-Slit Diffraction

Diffraction refers to the bending of waves around obstacles and apertures. Consider a slit of width a. The condition for minima in the diffraction pattern is

$$a\sin\theta = m\lambda, \quad m = \pm 1, \pm 2, \ldots$$

The central maximum is twice as wide as the secondary maxima. The intensity at angle θ is given by

$$I(\theta) = I_0\left(\frac{\sin(\beta)}{\beta}\right)^2, \quad \beta = \frac{\pi a\sin\theta}{\lambda}$$

11.5.5 Rayleigh's Criteria

Two point sources are said to be just resolved if the central maximum of one diffraction pattern coincides with the first minimum of the other. The Rayleigh criterion gives the limit at which two point sources can be resolved:

$$\theta_R = 1.22\frac{\lambda}{D}$$

Proof. Consider a circular aperture of diameter D. For monochromatic light of wavelength λ, the far-field (Fraunhofer) diffraction pattern of a point source is the **Airy pattern**, whose intensity is given by

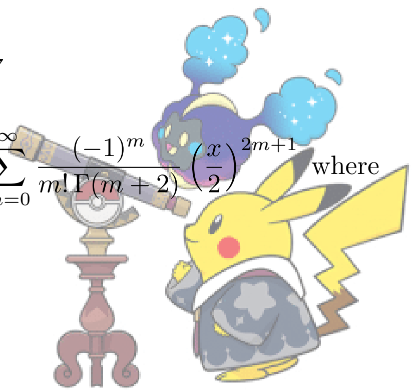
$$I(\theta) = I_0\left[\frac{2J_1(ka\sin\theta)}{ka\sin\theta}\right]^2$$

where $a = D/2$, $k = 2\pi/\lambda$, J_1 is called the Bessel function of the first kind of order 1[2], and θ is the angular distance from the optical axis.

The first zero of $J_1(x)$ occurs at $x \approx 3.8317$. Let $x = ka\sin\theta$. Then

$$ka\sin\theta_{\min} = 3.8317 \quad \Longrightarrow \quad \frac{2\pi a}{\lambda}\sin\theta_{\min} = 3.8317$$

[2] which is the solution to $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (x^2-1)y = 0$ in the form of $J_1(x) = \displaystyle\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma(m+2)}\left(\frac{x}{2}\right)^{2m+1}$ where $\Gamma(z) = \displaystyle\int_0^{\infty} x^{z-1}e^{-x}\,dx$ for $\Re(z) > 0$



Hence,

$$\sin\theta_{\text{min}} \approx \frac{3.8317\lambda}{2\pi(D/2)} \approx \frac{1.22\lambda}{D}$$

For small angles (θ in radians),

$$\theta_{\text{min}} \approx \frac{1.22\lambda}{D}$$

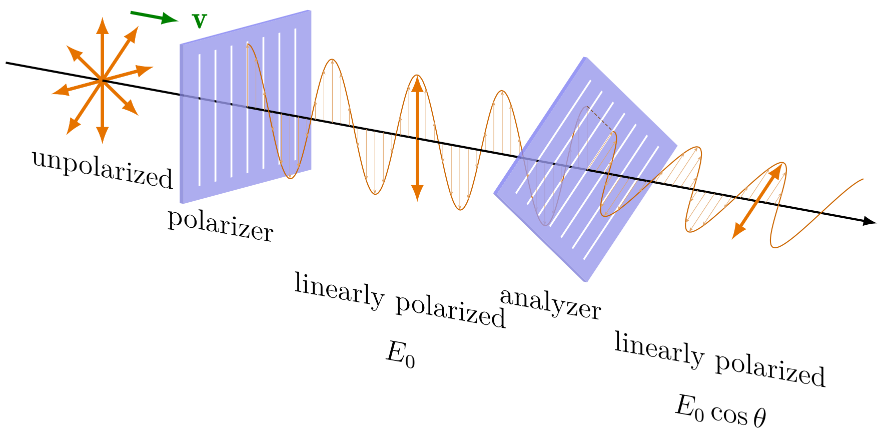
This θ_{min} is the angular radius of the Airy disk (to the first dark ring).

11.6 Polarization

11.6.1 Introduction

Polarization describes the direction in which the electric field vector of a light wave oscillates. Light can be

- **Unpolarized:** The electric field points in random directions (like sunlight).
- **Linearly polarized:** The electric field oscillates in a single direction.
- **Circularly polarized:** The electric field rotates in a circle as light travels.
- **Elliptically polarized:** A general case where the electric field traces an ellipse.

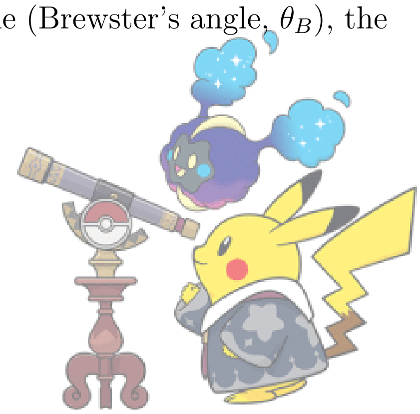


There are some methods for polarization:

- **Polarization by Reflection:** When light is reflected at a certain angle (Brewster's angle, θ_B), the reflected light becomes completely polarized:

$$\tan\theta_B = \frac{n_2}{n_1}$$

  where n_1 and n_2 are refractive indices of the two media.



- **Polarization by Absorption:** Polarizing filters only allow electric field components in a particular direction to pass through. When linearly polarized light passes through a polarizing filter, the transmitted intensity I is given by

$$I = I_0 \cos^2\theta \text{ (Malus's Law)}$$

  where

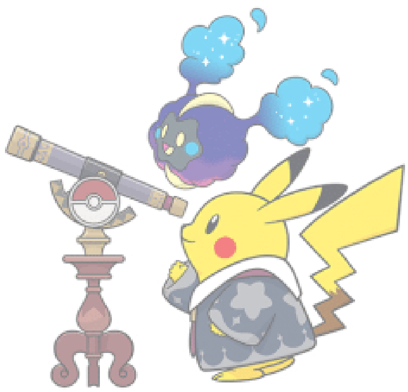
  - I_0 is the initial intensity of the light.
  - θ is the angle between the light's polarization direction and the axis of the polarizer.

11.6.2 Faraday's Rotation

The Faraday effect, or Faraday rotation, is a magneto-optic phenomenon where the plane of polarization of linearly polarized light is rotated when the light propagates through a material subjected to a strong, static magnetic field aligned in the direction of propagation. This effect is one of the first historical pieces of evidence linking light with electromagnetism.

When linearly polarized light passes through a transparent material of length L that is immersed in a magnetic field $\mathbf{B}$ (parallel to the direction of propagation), the angle of rotation β of the polarization plane is given by

$$\beta = VBL$$



12 Quantum Mechanics

12.1 Atomic Structure

12.1.1 Historical Development

- **Thomson's Model (1897):** "Plum pudding" model with electrons embedded in positive charge
- **Rutherford's Model (1911):** Nuclear model from alpha scattering experiments
- **Bohr's Model (1913):** Quantized electron orbits
- **Quantum Mechanical Model:** Electron clouds/orbitals

12.1.2 Introduction

The atom consists of

- **Nucleus:** Protons (p^+) and neutrons (n)
- **Electrons:** Negatively charged particles in orbitals

Particle	Symbol	Charge	Mass (u)
Proton	p or ${}^1_1$H	+1	1.007276
Neutron	n	0	1.008665
Electron	e^-	-1	0.000549

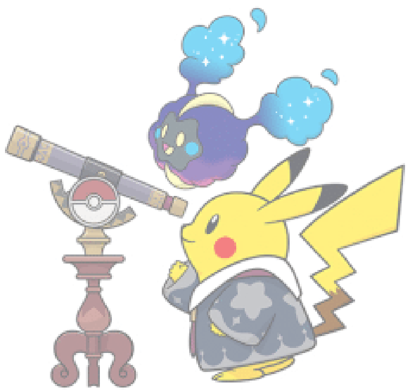
Table 4: Basic atomic particles

For an element X,

$${}^A_Z X$$

where

- A is the mass number (protons + neutrons),
- Z is the atomic number (number of protons), and
- we can define N to be neutron number ($N = A - Z$)



Example. ${}^{12}_{6}C_6$ has 6 protons and 6 neutrons.

The actual mass of a nucleus is **less** than the sum of its constituent particles:

$$\Delta m = (Zm_p + Nm_n) - m_{\text{nucleus}}$$

where

- Δm represents the mass defect,
- m_p represents the mass of proton,
- m_n represents the mass of neutron, and
- m_{nucleus} represents the measured nuclear mass.

According to Einstein's mass-energy equivalence

$$E = mc^2$$

The binding energy is

$$BE = \Delta m \cdot c^2$$

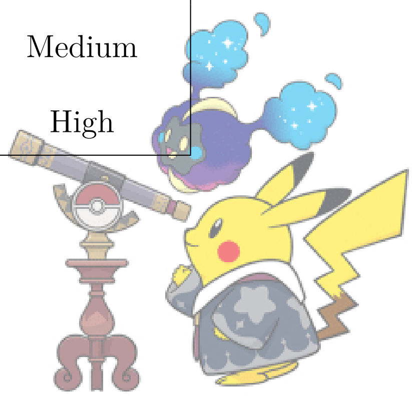
More conveniently, using atomic mass units (u) where $1\,u = 931.5\,\text{MeV}/c^2$:

$$BE\,(\text{MeV}) = \Delta m\,(u) \times 931.5$$

12.1.3 Nuclear Decay

Type	Emitted Particle	Change in Nucleus	Penetration
Alpha (α)	${}^{4}_{2}He$ nucleus	$Z \to Z-2,\ A \to A-4$	Low
Beta (β^-)	Electron (e^-)	$n \to p + e^- + \bar{\nu}$	Medium
Beta (β^+)	Positron (e^+)	$p \to n + e^+ + \nu$	Medium
Gamma (γ)	Photon (γ)	No change in Z or A	High

Table 5: Types of radioactive decay



Radioactive decay follows first-order kinetics:

$$N(t) = N_0 e^{-\lambda t}$$

where

- $N(t)$ represents the number of nuclei at time t,
- N_0 represents the initial number of nuclei, and
- $\lambda = \dfrac{\ln 2}{t_{1/2}}$ represents the decay constant where $t_{1/2}$ is half-life (time for half the nuclei to decay).

12.1.4 Neutrinos

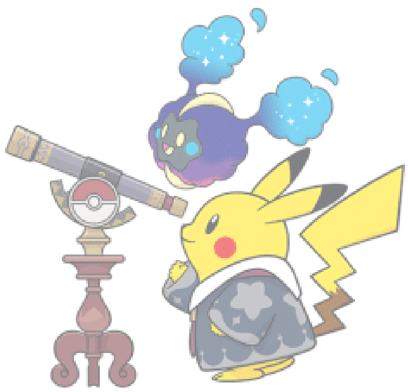
Introductionn Neutrinos are elementary particles that are part of the lepton family. They are electrically neutral and have an extremely small mass, making them very difficult to detect. Neutrinos interact only through weak nuclear force and gravity, which is why they pass through matter almost unaffected. The three types of neutrinos correspond to their associated charged leptons:

- Electron neutrino (ν_e)
- Muon neutrino (ν_μ)
- Tau neutrino (ν_τ)

These particles are produced in various high-energy processes, such as nuclear reactions in the Sun and other stars, as well as in cosmic ray interactions and supernovae.

Solar Neutrinos The Sun is a primary source of neutrinos, particularly electron neutrinos (ν_e). These solar neutrinos are produced during nuclear fusion processes that take place in the Sun's core. The dominant fusion reaction in the Sun is the proton-proton chain, which is responsible for the majority of the energy production. In this process, four protons fuse to form a helium nucleus, releasing energy in the form of gamma rays, neutrinos, and positrons.
The overall process can be written as

$$4\,p \longrightarrow {}^4\text{He} + 2\,e^+ + 2\,\nu_e + \gamma$$



12.2 Wave-Particle Duality

Louis de Broglie proposed that all matter has wave-like properties. The de Broglie wavelength is given by

$$\lambda = \frac{h}{p}$$

where

- λ is the de Broglie wavelength,
- $h = 6.626 \times 10^{-34}$ J s is Planck's constant, and
- p is the particle's momentum.

12.3 Planck's Equation

For electromagnetic waves of frequency ν, the energy of photon is given by

$$E = h\nu$$

12.4 Bohr Model of the Hydrogen Atom

12.4.1 Bohr Postulates

1. The electron moves in circular orbits around the proton due to Coulomb attraction.
2. Only certain discrete orbits are allowed.
3. The angular momentum of the electron is quantized.
4. Radiation is emitted or absorbed only during transitions between allowed orbits.

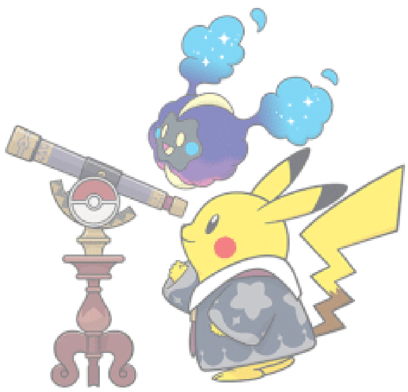
12.4.2 Formulas

The electrostatic force between the proton and electron is

$$F = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}$$

For circular motion, the centripetal force is

$$F = \frac{mv^2}{r}$$



Equating the two,

$$\frac{mv^2}{r} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}$$

Solving for velocity,

$$v^2 = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{mr}$$

Bohr postulated

$$L = mvr = n\hbar \quad n = 1, 2, 3, \ldots$$

Solving for velocity,

$$v = \frac{n\hbar}{mr}$$

Substitute the expression for v:

$$\left(\frac{n\hbar}{mr}\right)^2 = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{mr}$$

Solving for r,

$$r_n = \frac{4\pi\varepsilon_0\hbar^2}{me^2}n^2$$

Define the Bohr radius

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{me^2}$$

Hence,

$$r_n = a_0 n^2$$

The total energy is

$$E = K + U = \frac{1}{2}mv^2 - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}$$

Note that

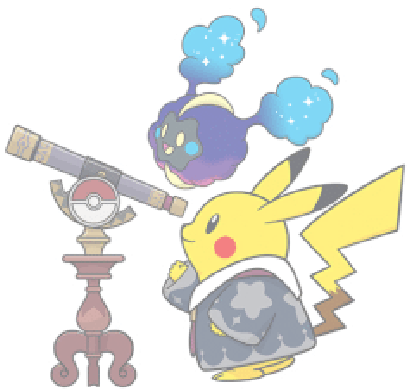
$$K = \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}$$

Hence, the total energy becomes

$$E = -\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}$$

Substituting r_n,

$$E_n = -\frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2}\frac{1}{n^2} = \frac{-13.6 \text{ eV}}{n^2}$$



12.4.3 Wavefunction

In quantum mechanics, the state of a particle is fully described by a complex-valued function called the **wavefunction**, denoted by $\psi(x,t)$. The wavefunction contains all the measurable information about the system.

The physical interpretation of the wavefunction is given by the Born rule, which states that the probability density of finding a particle at position x at time t is

$$P(x,t) = |\psi(x,t)|^2$$

Since the particle must be found somewhere in space, the total probability of finding the particle over all space must be equal to one. This requirement leads to the **normalisation condition**:

$$\int_{-\infty}^{\infty} |\psi(x,t)|^2 \, dx = 1$$

If a wavefunction does not initially satisfy this condition, it can be normalised by introducing a constant A such that

$$\Psi(x,t) = A\psi(x,t)$$

where the constant A is chosen to ensure the normalisation condition is satisfied.

12.4.4 Time-Independent Schrödinger Equation

The Schrödinger equation is

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(r)\psi = E\psi$$

with the Coulomb potential

$$V(r) = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}$$

Using separation of variables,

$$\psi(r,\theta,\phi) = R(r)Y_{\ell m}(\theta,\phi)$$

the radial equation becomes

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left[\frac{2m}{\hbar^2}\left(E + \frac{e^2}{4\pi\varepsilon_0 r}\right) - \frac{\ell(\ell+1)}{r^2}\right]R = 0$$

For large r, the Coulomb term becomes negligible, yielding

$$\frac{d^2R}{dr^2} - \kappa^2 R = 0$$

where

$$\kappa = \sqrt{-\frac{2mE}{\hbar^2}}$$

The physically acceptable solution is

$$R(r) \sim e^{-\kappa r}$$

For small r, the equation has solution

$$R(r) \sim r^\ell$$

Motivated by the asymptotic behavior, write

$$R(r) = r^\ell e^{-\kappa r} F(r)$$

Define a dimensionless variable

$$\rho = 2\kappa r$$

and substitute into the radial equation to obtain

$$\rho \frac{d^2F}{d\rho^2} + (2\ell + 2 - \rho)\frac{dF}{d\rho} + \left(\frac{me^2}{2\pi\varepsilon_0\hbar^2\kappa} - \ell - 1\right) F = 0$$

Assume a power series

$$F(\rho) = \sum_{k=0}^{\infty} a_k \rho^k$$

Substitution yields the recursion relation

$$a_{k+1} = \frac{k + \ell + 1 - \frac{me^2}{2\pi\varepsilon_0\hbar^2\kappa}}{(k+1)(k+2\ell+2)} a_k$$

For the wavefunction to remain normalizable, the series must terminate. Hence, there exists an integer n_r such that

$$\frac{me^2}{2\pi\varepsilon_0\hbar^2\kappa} = n_r + \ell + 1$$

Define the principal quantum number

$$n = n_r + \ell + 1$$

Solving for energy,

$$E_n = -\frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2}\frac{1}{n^2}$$

12.5 Rydberg Formula

The wavelength λ of the emitted or absorbed light in a hydrogen atom is given by

$$\frac{1}{\lambda} = R_\infty\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$

where

- λ is the wavelength of the light,
- R_∞ is the Rydberg constant for hydrogen, approximately $R_\infty = 1.097 \times 10^7\,\mathrm{m}^{-1}$, and
- n_1 and n_2 are positive integers, with $n_2 > n_1$.

Proof. According to the Bohr model, the energy levels of a hydrogen atom are quantized and given by

$$E_n = -\frac{13.6\ \mathrm{eV}}{n^2}$$

When an electron transitions from a higher energy level n_2 to a lower energy level n_1, the energy difference ΔE is given by

$$\Delta E = E_{n_1} - E_{n_2} = -\frac{13.6\ \mathrm{eV}}{n_1^2} + \frac{13.6\ \mathrm{eV}}{n_2^2}$$

The energy of the emitted photon when the electron undergoes a transition is

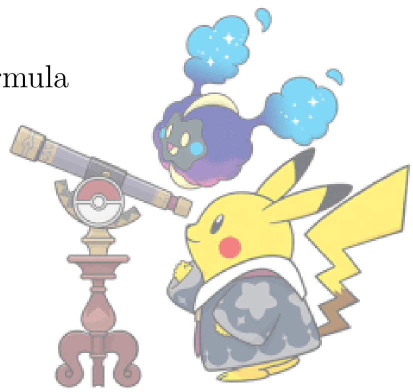
$$E_{\text{photon}} = \Delta E = h\nu$$

The frequency ν of the emitted photon is related to the wavelength λ by

$$\nu = \frac{c}{\lambda}$$

By combining the expressions for E_{photon} and ΔE, we obtain the Rydberg formula

$$\frac{1}{\lambda} = R_\infty\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$



12.6 Uncertainty Principle

Werner Heisenberg showed that certain pairs of physical quantities cannot be simultaneously measured with arbitrary precision. The most famous uncertainty relation is between position and momentum:

$$\Delta x \, \Delta p \geq \frac{\hbar}{2}$$

where

- Δx is the uncertainty in position
- Δp is the uncertainty in momentum
- $\hbar = \dfrac{h}{2\pi}$ is the reduced Planck constant

Example. Consider a particle (e.g., an electron) passing through a narrow slit of width a along the x-direction. Due to the confinement in position, the particle's wavefunction is localized within the slit, giving

$$\Delta x \sim a$$

After passing through the slit, the particle undergoes diffraction, resulting in a spread in its momentum in the x-direction. The diffraction pattern for a slit is given by the first minimum condition:

$$a \sin\theta \sim \lambda$$

where θ is the diffraction angle and λ is the wavelength of the particle. For a particle with momentum p, its wavelength is related to the momentum by de Broglie's relation:

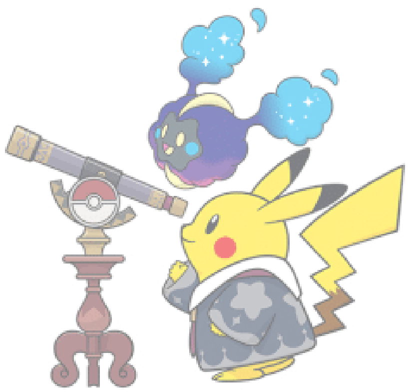
$$\lambda = \frac{h}{p}$$

where h is Planck's constant. The transverse momentum uncertainty is approximately

$$\Delta p_x \sim p \sin\theta \sim \frac{h}{a}$$

Multiplying the uncertainties in position and momentum:

$$\Delta x \, \Delta p_x \sim a \cdot \frac{h}{a} \sim h \sim \hbar$$

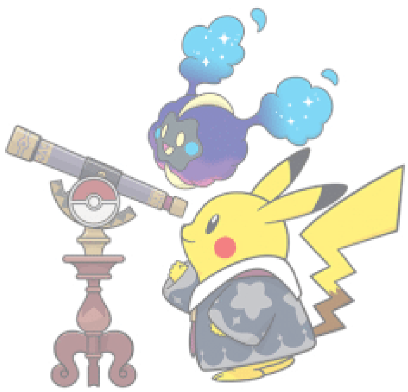


which reproduces the Heisenberg Uncertainty Principle approximately.

Another important uncertainty relation is between energy and time:

$$\Delta E \, \Delta t \geq \frac{\hbar}{2}$$

The uncertainty principle isn't about measurement limitations, but about fundamental properties of quantum systems. Particles don't have precisely defined positions and momenta simultaneously.



13 Stellar Astrophysics

13.1 Stellar Classifications

Stars can be classified by their internal structure:

- **Main Sequence Stars**: Hydrogen-burning core, radiative or convective energy transport. Governed by mass-luminosity relation:

$$\frac{L}{L_\odot} \approx \begin{cases} \left(\frac{M}{M_\odot}\right)^{4.0} & \text{for } M > 10M_\odot \\ \left(\frac{M}{M_\odot}\right)^{3.5} & \text{for } 0.5 < M < 10M_\odot \\ \left(\frac{M}{M_\odot}\right)^{2.3} & \text{for } M < 0.5M_\odot \end{cases}$$

- **Giant Stars**: Expanded envelope, hydrogen shell burning around inert helium core.
- **Supergiants**: Massive stars with complex layered burning (H, He, C, O, Si shells).
- **White Dwarfs**: Electron-degenerate cores, supported by electron degeneracy pressure.
- **Neutron Stars**: Neutron-degenerate matter, extreme density ($\sim 10^{17}$ kg/m^3).
- **Black Holes**: Gravitational collapse beyond neutron degeneracy pressure support.

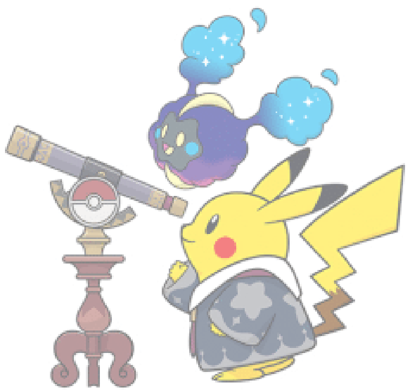
Stars can also be classified by their composition:

Population	**Metallicity [Fe/H]**	**Characteristics**
Population I	≥ -0.5	Metal-rich, disk stars, young
Population II	-1.0 to -0.5	Metal-poor, halo stars, old
Population III	$\ll -1.0$	Zero metals, first generation

Table 6: Stellar Populations

Metallicity affects stellar evolution:

- Opacity changes with metal content
- Lower metallicity stars are hotter and bluer at given mass



- Metallicity influences mass loss rates

Stars can also be classified by their activity:

- **Flare Stars** (UV Ceti type): M-dwarfs with magnetic reconnection events: Magnetic reconnection is a fundamental plasma physics process where the topology of magnetic field lines is rearranged, converting magnetic energy into kinetic energy, thermal energy, and particle acceleration.
- **Rotational Classes**:
  - Slow rotators: $v_{\text{eq}} < 10$ km/s (Sun: 2 km/s)
  - Fast rotators: $v_{\text{eq}} > 50$ km/s (young stars)
- **Magnetic Activity**:

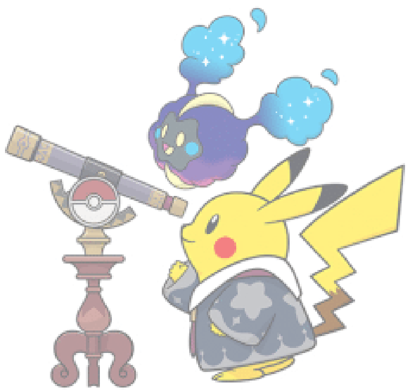
$$R'_{HK} = \frac{L_{HK}}{L_{\text{bol}}}$$

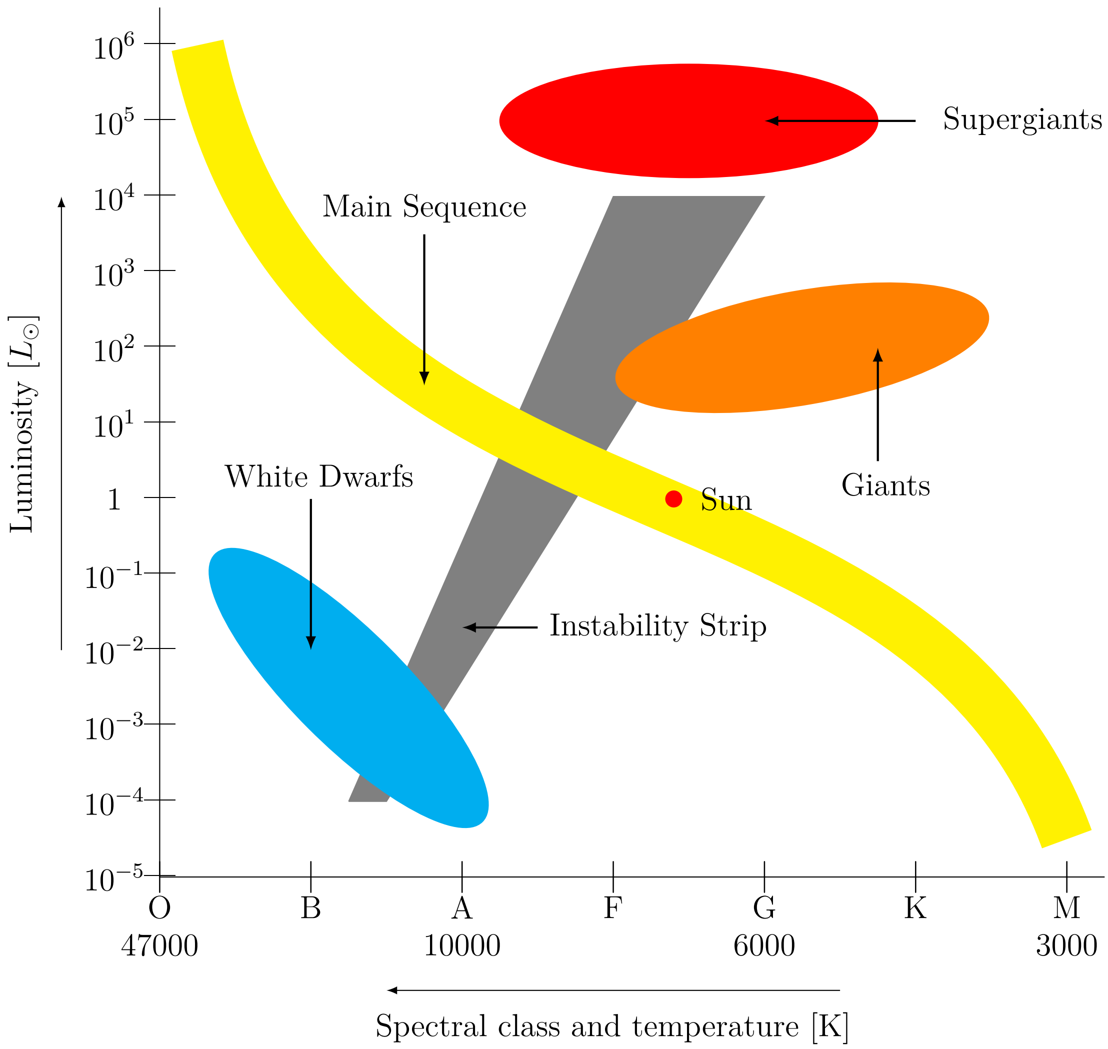
  where L_{HK} is the luminosity in the Calcium II H & K lines and L_{bol} is the bolometric luminosity: Total power output across all wavelengths (from radio to X-rays). This ratio tells what fraction of the star's total energy output is being emitted specifically from its magnetically heated chromosphere via these Calcium lines as Calcium II is singly-ionized calcium.

13.2 HR Diagram

13.2.1 Introduction

The Hertzsprung-Russell diagram (HR diagram) is one of the most important tools in astrophysics, independently developed by Ejnar Hertzsprung (1905-1911) and Henry Norris Russell (1913). It revolutionized our understanding of stellar evolution by revealing patterns in stellar properties.





13.2.2 Spectral Types and Temperature

Each spectral type corresponds to a particular range of temperatures and characteristics. The order from hottest to coolest stars is as follows:

Spectral Type	**Temp (K)**	**Color**	**Mass ($M_\odot$)**	**Examples**
O	30,000-50,000	Blue	15-90	ζ Puppis
B	10,000-30,000	Blue-white	2-15	Rigel, Spica
A	7,500-10,000	White	1.4-2	Sirius, Vega
F	6,000-7,500	Yellow-white	1.04-1.4	Procyon
G	5,200-6,000	Yellow	0.8-1.04	Sun, α Cen A
K	3,700-5,200	Orange	0.45-0.8	Arcturus, Aldebaran
M	2,400-3,700	Red	0.08-0.45	Betelgeuse, Proxima Cen

Table 7: MK Spectral Classification System

The sequence ”O, B, A, F, G, K, M” can be difficult to remember due to the variety of letters. To aid in memorization, astronomers and students often use mnemonics. A common mnemonic for remembering the spectral types in order is

“Oh Be A Fine Girl/Guy, Kiss Me!”

This mnemonic associates each letter with a word:

- **O** - Oh
- **B** - Be
- **A** - A
- **F** - Fine
- **G** - Girl/Guy
- **K** - Kiss
- **M** - Me

In addition to spectral types, stars are also classified based on their luminosity, which is related to their size and brightness. This is done using Roman numerals from I to V:

- **I**: Supergiants (e.g., Betelgeuse)

- **II**: Bright giants
- **III**: Giants (e.g., Alpha Centauri)
- **IV**: Subgiants
- **V**: Main sequence stars (e.g., the Sun)

The full classification of a star would be a combination of both spectral type and luminosity class. For example, our Sun is classified as a **G2V** star, meaning it is a G-type main sequence star. 2 is a subclass number that further refines the classification. The scale goes from 0 to 9, with 0 being the hottest and 9 being the coolest in each spectral type. The number 2 indicates that the star is towards the middle of the G-type range. In this case, a G2 star has a temperature closer to $5,800\,\mathrm{K}$.

13.2.3 Turn-Off Point

The **turn-off point** is an important feature in the HR diagram that helps astronomers determine the age of a star cluster. It represents the point where stars, after exhausting the hydrogen in their cores, begin to leave the main sequence and evolve into red giants. The position of the turn-off point on the diagram depends on the mass of the stars in the cluster.
Note that

$$\text{Age of Cluster} = \frac{M}{L} \propto \frac{M}{M^{3.5}} = \frac{1}{M^{2.5}}$$

where we use the mass of the star at the turn-off point M.

13.3 Stellar Evolution

13.3.1 Stellar Formation

Stars form from giant clouds of gas and dust called **molecular clouds** or **nebulae**. The process includes:

- **Gravitational Collapse:** Regions with higher density collapse under gravity.
- **Protostar Formation:** As the cloud collapses, a dense core forms and heats up due to gravitational energy.
- **Accretion:** Surrounding material falls onto the protostar, increasing its mass and temperature.

13.3.2 Pre-Main Sequence Stars

Before reaching the main sequence, stars go through the **pre-main sequence (PMS)** phase:

- The protostar contracts and heats up.
- Energy is produced mainly by gravitational contraction, not nuclear fusion.
- PMS stars follow **Hayashi tracks** (for lower-mass stars) or **Henyey tracks** (for higher-mass stars) on the Hertzsprung-Russell diagram.

13.3.3 Main Sequence Stars

A star enters the **main sequence** when hydrogen fusion begins in its core.

- Core hydrogen fusion converts hydrogen into helium via the **proton-proton chain** (low-mass stars) or **CNO cycle** (high-mass stars).
  - **Proton-Proton Chain**:

$$4\,^1\text{H} \longrightarrow \,^4\text{He} + 2e^+ + 2\nu_e + 26.7\,\text{MeV}$$

  - **CNO Cycle**:

$$^{12}\text{C} \longrightarrow \,^1\text{H} \longrightarrow \,^4\text{He} + \text{energy}$$

- The star achieves **hydrostatic equilibrium**: gravity balanced by radiation pressure from fusion.
- The main sequence lifetime depends on stellar mass; higher-mass stars burn faster and live shorter lives.

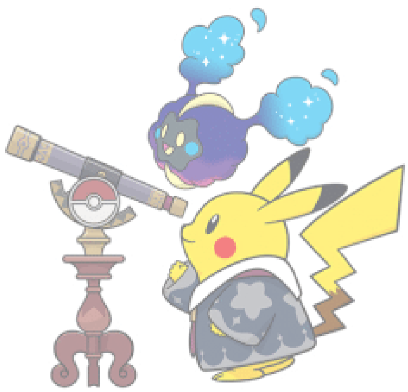
13.3.4 Post-Main Sequence Stars

After hydrogen in the core is exhausted, stars evolve differently depending on their mass.

Low to Intermediate-Mass Stars ($< 8M_\odot$)

- Expand into **Red Giants**.
- Helium fusion begins in the core (**triple-alpha process**):

$$3\,^4\text{He} \longrightarrow \,^{12}\text{C} + \gamma$$



- May go through **asymptotic giant branch (AGB)** phase before shedding outer layers.

High-Mass Stars ($> 8M_\odot$)

- Expand into **supergiants**.
- Fuse heavier elements in successive shells (C, O, Si, etc.).
- Form an **iron core**, which cannot undergo fusion to produce energy.

13.3.5 Supernovae

- Massive stars end their lives in a **core-collapse supernova** (Type II).
- The core collapses, and outer layers are expelled violently.
- Supernovae enrich the interstellar medium with heavy elements.

13.3.6 Planetary Nebulae

- Formed by low to intermediate-mass stars shedding their outer layers.
- The exposed core emits ultraviolet radiation, ionizing the ejected gas.
- Eventually fades to leave a **white dwarf**.

13.3.7 End States of Stars

- **White Dwarfs:** Remnants of low/intermediate-mass stars. Supported by electron degeneracy pressure.
- **Neutron Stars:** Remnants of core-collapse supernovae of stars with $8 - 20M_\odot$. Supported by neutron degeneracy pressure.
- **Black Holes:** Remnants of very massive stars ($> 20M_\odot$). Gravity overwhelms all forms of degeneracy pressure.

13.4 Magnitude System

Definition. 13.1: Luminosity

Luminosity is the **total amount of energy emitted by a source per unit time**.

Definition. 13.2: Luminance

Luminance is the amount of **visible light emitted or reflected in a given direction per unit area per unit solid angle**. It describes how bright a surface appears to the human eye.

Property	Luminosity	Luminance
Physical meaning	Total power emitted	Brightness per area per direction
Depends on distance	No	No (but depends on direction)
Depends on area	Indirectly	Directly
SI unit	Watt (W)	cd/m^2
Usually used in	Astrophysics	Optics

Definition. 13.3: Flux

The **flux** (F or Φ) is the total energy received from an astronomical object per unit time per unit area. It represents the power (energy per unit time) crossing a unit area oriented perpendicular to the direction of propagation.

For a source emitting total luminosity L isotropically, the flux measured at distance r is

$$F = \frac{L}{4\pi r^2}$$

Definition. 13.4: Solid Angle

The **solid angle** measures the three-dimensional angular size of an object as seen from a point. It is defined as

$$d\Omega = \frac{dA}{r^2}$$

where

- dA is an area element on a sphere of radius r,
- $d\Omega$ is the corresponding solid angle.

Definition. 13.5: Intensity

The flux per unit solid angle is called **intensity**:

$$I = \frac{dF}{d\Omega}$$

Definition. 13.6: Spectral Flux Density

The **spectral flux density** (F_λ or F_ν) is the flux per unit wavelength or frequency interval. It describes how the flux is distributed across the electromagnetic spectrum.

The **solar constant** is the flux received from the Sun at Earth's distance:

$$F_\odot = 1361 \text{ W m}^{-2} = 1.361 \times 10^6 \text{ erg s}^{-1}\text{cm}^{-2}$$

This represents the total solar power (all wavelengths) incident on 1 m^2 at 1 AU.

Definition. 13.7: Apparent Magnitude

It refers to the brightness of an object **as observed from Earth**, given by

$$m = -2.5 \log_{10}\left(\frac{F}{F_0}\right)$$

where F_0 is the reference flux for zero magnitude. The zero-point F_0 is calibrated using standard stars. Originally, Vega (α Lyrae) was defined to have magnitude 0.0 in all filters. Modern systems use more precisely defined spectrophotometric standards.

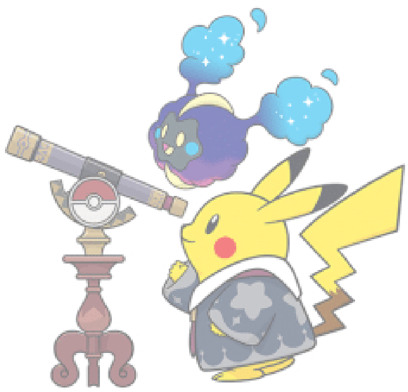
Ancient astronomers (especially Hipparchus, 150 BCE) ranked stars by eye:

1. 1st magnitude means the brightest (e.g. Sirius and Vega)

2. 6th magnitude means faintest visible to the naked eye

Much later (1856), Norman Pogson put this on a mathematical footing. He defined the scale so that a difference of 5 magnitudes corresponds to a brightness ratio of 100.

$$\frac{F_1}{F_2} = 100 \text{ when } m_2 - m_1 = 5$$



Assume magnitude difference is proportional to the logarithm of the flux ratio:

$$m_2 - m_1 = k \log_{10}\left(\frac{F_1}{F_2}\right)$$

By Pogson's condition, $5 = 2k \implies k = 2.5$. Take star 1 as the reference star: $m_1 = 0, F_1 = F_0$ gives $m = -2.5 \log_{10}\left(\frac{F}{F_0}\right)$. Later, the logarithmic scale was found to match the Weber-Fechner law in psychophysics, which states that perceived sensation is proportional to the logarithm of stimulus intensity.

Definition. 13.8: Absolute Magnitude

The brightness an object would have if placed at a standard distance of **10 parsecs** (32.6 light-years):

$$M = m - 5 \log_{10}\left(\frac{d}{10 \text{ pc}}\right)$$

where d is the distance to the object in parsecs.

Definition. 13.9: Distance Modulus

The difference between apparent and absolute magnitude relates directly to distance:

$$\mu = m - M = 5 \log_{10} d - 5$$

Definition. 13.10: Limiting Magnitude

The limiting magnitude is the faintest apparent magnitude of a celestial object that can be detected with a given instrument under specific observing conditions. It represents the threshold between what is observable and what is not. It is given by

$$m_{\text{lim}} = m_{\text{eye}} + 5 \log\left(\frac{D}{d_{\text{eye}}}\right) + 2.5 \log_{10}\left(\frac{\text{Transmission}}{0.95}\right)$$

where

1. m_{eye} is the naked-eye limiting magnitude (typically 6.0 under ideal conditions),
2. D is the telescope aperture,
3. d_{eye} is the dark-adapted pupil diameter (typically 7 mm), and
4. Transmission is the optical transmission coefficient.

Definition. 13.11: Bolometric Magnitude

The **bolometric magnitude** (m_{bol} or M_{bol}) is a measure of an astronomical object's total electromagnetic luminosity across all wavelengths, from gamma rays to radio waves. Unlike filter-specific magnitudes, it accounts for **all** emitted radiation.

The bolometric magnitude is defined through the **bolometric flux** F_{bol}, which is the integral of the spectral flux density over all wavelengths:

$$F_{\text{bol}} = \int_0^\infty F_\lambda \, d\lambda = \int_0^\infty F_\nu \, d\nu$$

The apparent bolometric magnitude is then

$$m_{\text{bol}} = -2.5 \log_{10}\left(\frac{F_{\text{bol}}}{F_{\text{bol},0}}\right)$$

where $F_{\text{bol},0}$ is the reference bolometric flux for zero magnitude.

Definition. 13.12: Bolometric Correction

It refers to the difference between bolometric and visual magnitudes:

$$BC = M_{\text{bol}} - M_V$$

or for apparent magnitudes:

$$BC = m_{\text{bol}} - m_V$$

The Sun's magnitude is assumed to be bolometric by convention. Therefore, it is common to use

$$M_{\text{bol}} - M_\odot = -2.5 \log_{10}\left(\frac{L}{L_\odot}\right).$$

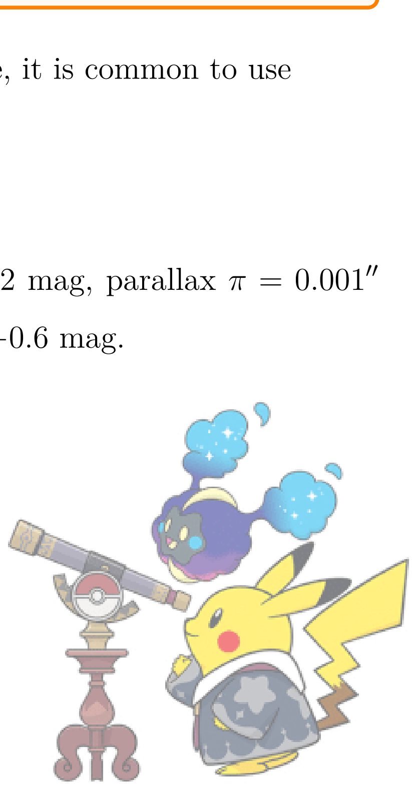
Example. **(2013 IOAA)** A star has visual apparent magnitude $m_V = 12.2$ mag, parallax $\pi = 0.001''$ and effective temperature $T_{\text{eff}} = 4000\,\text{K}$. Its bolometric correction is B.C. $= -0.6$ mag.

(a) Find its luminosity as a function of the solar luminosity.

(b) What type of star is it?

(i) a red giant

(ii) a blue giant



(iii) a red dwarf

Please write (i), (ii) or (iii) in your answer sheet.

(a) First, the absolute visual magnitude is obtained from

$$M_V - m_V = 5 - 5\log r$$

or equivalently,

$$M_V - m_V = 5 + 5\log \pi$$

Thus,

$$M_V = 12.2 + 5 + 5\log(0.001) = 12.2 + 5 - 15 = 2.2 \text{ mag}$$

The bolometric correction is defined as

$$\text{B.C.} = M_{\text{bol}} - M_V$$

so

$$M_{\text{bol}} = \text{B.C.} + M_V = -0.6 + 2.2 = 1.6 \text{ mag}$$

The luminosity is calculated using

$$M_\odot - M_{\text{bol}} = 2.5\log\left(\frac{L}{L_\odot}\right)$$

where $M_\odot = 4.72$ mag. Hence,

$$4.72 - 1.6 = 2.5\log\left(\frac{L}{L_\odot}\right)$$

$$\log\left(\frac{L}{L_\odot}\right) = 1.25$$

and therefore

$$L = 17.7\,L_\odot$$

(b) A star with $M_{\text{bol}} = 1.6$ mag, luminosity $L = 17.7\,L_\odot$, and effective temperature $T_{\text{eff}} = 4000\,\text{K}$ is much brighter and cooler than the Sun. Therefore, it is **(i) a red giant star**.

13.5 Albedo

Albedo is a dimensionless physical quantity that measures the reflectivity of a surface. It is defined as the fraction of incident electromagnetic radiation (usually sunlight) that is reflected by a surface.

If a surface receives an incident radiant power P_{in} and reflects a power P_{ref}, its albedo α is defined as

$$\alpha = \frac{P_{\text{ref}}}{P_{\text{in}}}, \qquad 0 \leq \alpha \leq 1$$

Assuming the planet radiates as a black body and is in thermal equilibrium, the absorbed power equals emitted power:

$$(1-\alpha)\pi R^2 S = 4\pi R^2 \sigma T^4$$

where σ is the Stefan–Boltzmann constant.

Solving for the equilibrium temperature T:

$$\boxed{T = \left(\frac{(1-\alpha)S}{4\sigma}\right)^{1/4}}$$

13.6 Geometric Albedo

The **geometric albedo** is a dimensionless quantity that measures how bright an astronomical body appears when observed at **full phase** (i.e. zero phase angle), compared to an idealized reference surface. Let

- α be the **phase angle**, defined as the angle between the incident radiation from the source (e.g. the Sun) and the direction to the observer, as seen from the object,
- $\alpha = 0$ correspond to **full illumination** (observer directly behind the light source).

We introduce a reference surface:

> A perfectly diffusing (Lambertian) flat disk with the same cross-sectional area as the object, illuminated and observed at normal incidence.

A Lambertian surface reflects radiation isotropically, obeying Lambert’s cosine law:

Theorem. 13.1: Lambert’ s Cosine law

A surface obeys **Lambert’s cosine law** if its radiance is independent of direction:

$$I(\theta, \phi) = I_0 = \text{constant}$$

For a Lambertian surface, the power emitted into solid angle $d\Omega$ is

$$dP = I_0 \cos\theta \, dA \, d\Omega$$

If the incident solar flux is F_{inc}, a perfectly reflecting disk intercepts power $P = F_{inc}A$. Integrating the Lambertian emission over a hemisphere:

$$P_{reflected} = \int_0^{2\pi} \int_0^{\pi/2} I_{ref} \cos\theta \sin\theta \, A \, d\theta \, d\phi = \pi A I_{ref}$$

Equating incident and reflected power ($P_{reflected} = F_{inc}A$), we find:

$$I_{ref} = \frac{F_{inc}}{\pi}$$

The geometric albedo p is defined as

$$p = \frac{I_{object}(\alpha = 0)}{I_{ref}} = \frac{\pi I_{object}(\alpha = 0)}{F_{inc}}$$

The total energy reflected in all directions is characterized by the **Bond Albedo** (A_B), related to p by the **phase integral** q:

$$A_B = p \cdot q, \quad q = 2 \int_0^{\pi} \Phi(\alpha) \sin\alpha \, d\alpha$$

where $\Phi(\alpha)$ is the phase function, which is the relative brightness of the object as a function of the phase angle, normalized such that $\Phi(0) = 1$.

13.7 Color Indices

Color indices quantify the color of astronomical objects by measuring the difference in magnitude between two different wavelength bands:

$$C = m_{\lambda_1} - m_{\lambda_2}$$

where m_{λ_1} and m_{λ_2} are apparent magnitudes measured through different filters.

Filter	**Center (λ_c, nm)**	**FWHM (nm)**	**Typical Use**
U	365	66	Ultraviolet continuum
B	445	94	Blue light, Balmer break
V	551	88	Visual (photopic)
R	658	138	Red continuum
I	806	149	Near-infrared

Table 8: Johnson-Cousins photometric system filters

The followings are the common color indices to use:

$$U - B = m_U - m_B$$
$$B - V = m_B - m_V$$
$$V - R = m_V - m_R$$
$$V - I = m_V - m_I$$

The observed color index is affected by interstellar extinction:

$$(B - V)_{\text{obs}} = (B - V)_0 + E(B - V)$$

where $E(B - V)$ is the color excess:

$$E(B - V) = A_B - A_V$$

where A_B is the extinction in B-band in mag and A_V is the extinction in V-band in mag.

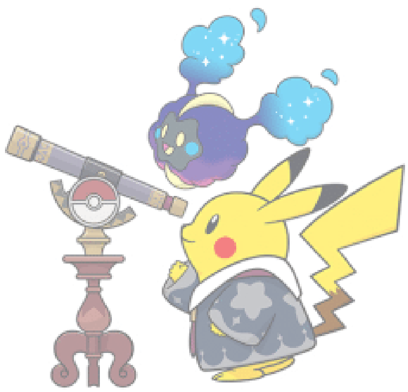
13.8 Atmospheric Extinction

Atmospheric extinction reduces the observed flux:

$$m_{\lambda,\text{obs}} = m_{\lambda,\text{true}} + k'_\lambda \cdot X$$

where

- k'_λ is the extinction coefficient (mag/airmass) and



- X is the airmass (dimensionless) which quantifies the atmosphere's thickness along a light's path.

For a plane-parallel atmosphere,

$$X = \sec z$$

where z is the zenith distance ($z = 90° - \text{altitude}$).

13.9 Optical Depth

Optical depth τ_ν describes attenuation by comparing the initial intensity $I_\nu(0)$ and the transmitted intensity $I_\nu(s)$:

$$\tau_\nu(s) \equiv -\ln\left(\frac{I_\nu(s)}{I_\nu(0)}\right)$$

Extinction in magnitudes A_λ is related to optical depth by

$$A_\lambda = 2.5\log_{10}(e\tau_\lambda) = 1.086\,\tau_\lambda$$

as

$$\frac{I}{I_0} = e^{-\tau_\lambda} = 10^{-0.4A_\lambda}$$

13.10 Binary Star

Binary star systems consist of two stars orbiting around their common center of mass.

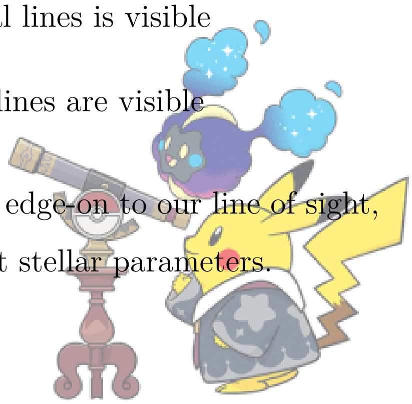
13.10.1 Different Types of Binary Stars

Visual Binaries They refer to the stars that can be resolved individually through telescopes. Their orbits can be directly observed over time.

Spectroscopic Binaries They refer to the stars that can be detected through periodic Doppler shifts in their spectral lines. They can be further identified as

- **Single-lined spectroscopic binaries (SB1):** Only one set of spectral lines is visible
- **Double-lined spectroscopic binaries (SB2):** Both sets of spectral lines are visible

Eclipsing Binaries They are the systems where the orbital plane is nearly edge-on to our line of sight, causing periodic eclipses. These provide the most complete information about stellar parameters.



Astrometric Binaries They can be detected through the wobble of one star's proper motion due to an unseen companion.

Interacting Binaries

- Mass transfer occurs when one star fills its Roche lobe.
- Examples: Cataclysmic variables, X-ray binaries.

Peculiar Binary Systems

- Systems with unusual properties such as extremely short orbital periods or highly eccentric orbits.
- Examples: Contact binaries, Algol-type binaries.

13.10.2 Modified Kepler's Third Law

For a binary system, Kepler's third law relates the orbital period P, semi-major axis a, and total mass M:

$$P^2 = \frac{4\pi^2 a^3}{G(M_1 + M_2)} \tag{28}$$

where

- P is the orbital period,
- a is the semi-major axis of the relative orbit, and
- M_1, M_2 are the masses of the two stars

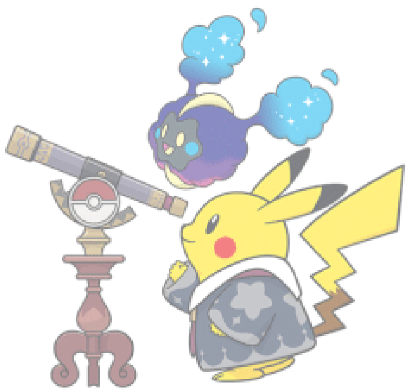
13.10.3 Mass Function

For spectroscopic binaries, we measure the **mass function**. For single-lined spectroscopic binaries,

$$f(M) = \frac{M_2^3 \sin^3 i}{(M_1 + M_2)^2} = \frac{P}{2\pi G} v_{1,r}^3$$

For double-lined spectroscopic binaries, we can determine the mass ratio:

$$\frac{M_1}{M_2} = \frac{v_{2,r}}{v_{1,r}}$$



Proof. We begin with Kepler's third law for two bodies orbiting their common center of mass:

$$P^2 = \frac{4\pi^2 a^3}{G(M_1 + M_2)}$$

where $a = a_1 + a_2$ is the total separation between the stars, and a_1, a_2 are their distances from the center of mass.

The center of mass condition gives

$$a_1 M_1 = a_2 M_2$$

From this, we can write the mass ratio:

$$\frac{a_1}{a_2} = \frac{M_2}{M_1} \tag{29}$$

Also, the total separation can be expressed in terms of a_1:

$$\begin{aligned} a &= a_1 + a_2 && (30) \\ &= a_1 + a_1 \frac{M_1}{M_2} \quad \text{(from Equation 29)} && (31) \\ &= a_1 \left(\frac{M_1 + M_2}{M_2} \right) && (32) \end{aligned}$$

For the visible star (star 1), we measure its **radial velocity amplitude** $v_{1,r}$. The true orbital speed v_1 is related to the observed radial velocity by the inclination:

$$v_{1,r} = v_1 \sin i \tag{33}$$

For a circular orbit ($e = 0$), the orbital speed is constant and given by

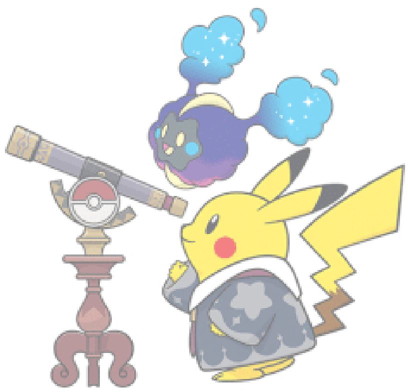
$$v_1 = \frac{2\pi a_1}{P} \tag{34}$$

Combining Equations 33 and 34:

$$v_{1,r} = \frac{2\pi a_1 \sin i}{P} \tag{35}$$

From Equation 35, we can solve for a_1:

$$a_1 = \frac{P v_{1,r}}{2\pi \sin i}$$



Now substitute Equation 32 into Kepler's law (Equation 13.10.3):

$$P^2 = \frac{4\pi^2}{G(M_1+M_2)}\left[a_1\left(\frac{M_1+M_2}{M_2}\right)\right]^3 \tag{36}$$

$$= \frac{4\pi^2 a_1^3}{G(M_1+M_2)}\cdot\frac{(M_1+M_2)^3}{M_2^3} \tag{37}$$

$$= \frac{4\pi^2 a_1^3}{G}\cdot\frac{(M_1+M_2)^2}{M_2^3} \tag{38}$$

Now substitute a_1 from Equation 13.10.3 into Equation 38:

$$P^2 = \frac{4\pi^2}{G}\cdot\frac{(M_1+M_2)^2}{M_2^3}\cdot\left(\frac{Pv_{1,r}}{2\pi\sin i}\right)^3$$

$$= \frac{4\pi^2}{G}\cdot\frac{(M_1+M_2)^2}{M_2^3}\cdot\frac{P^3v_{1,r}^3}{8\pi^3\sin^3 i}$$

$$= \frac{P^3v_{1,r}^3}{2\pi G\sin^3 i}\cdot\frac{(M_1+M_2)^2}{M_2^3}$$

Cancel P^2 from both sides (multiply both sides by $1/P^2$):

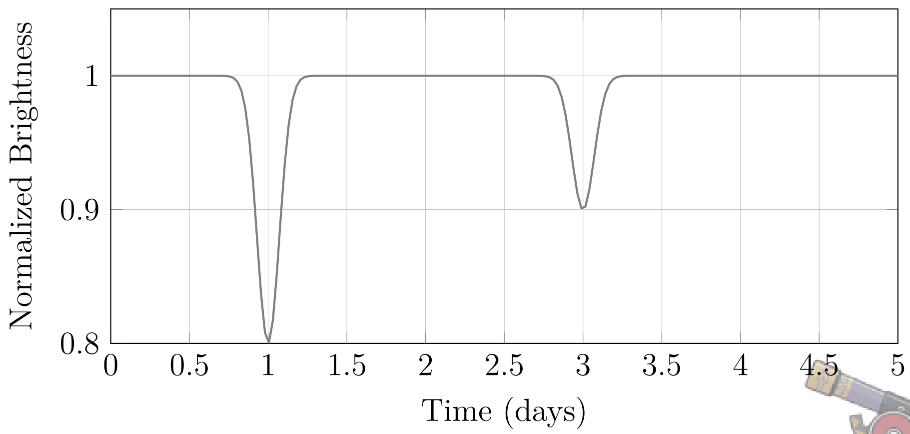
$$1 = \frac{Pv_{1,r}^3}{2\pi G\sin^3 i}\cdot\frac{(M_1+M_2)^2}{M_2^3}$$

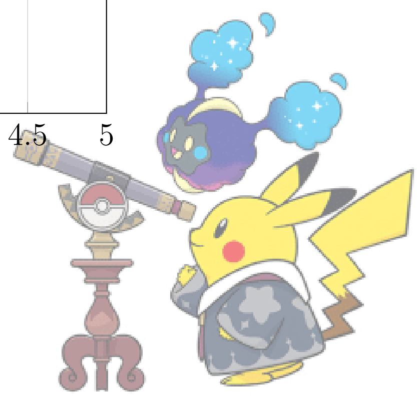
Now rearrange to isolate the mass terms on the left:

$$\frac{M_2^3\sin^3 i}{(M_1+M_2)^2} = \frac{P}{2\pi G}v_{1,r}^3$$

13.10.4 Light Curves

Eclipsing binaries exhibit characteristic light curves with periodic dips in brightness:





1. Let F_1 and F_2 be the flux of the two stars. The total observed flux when both stars are visible is

$$F_{\text{total}} = F_1 + F_2$$

2. The **primary eclipse** occurs when the brighter star is partially or fully blocked by the dimmer star, causing a significant dip. Let the obscured fraction of the primary star be $f_1(t)$, then the observed flux is

$$F_{\text{primary}}(t) = F_1\,[1 - f_1(t)] + F_2$$

3. The **secondary eclipse** occurs when the dimmer star is blocked, causing a smaller dip. The fainter star is obscured by the brighter star. Let $f_2(t)$ be the obscured fraction of the secondary star, then:

$$F_{\text{secondary}}(t) = F_1 + F_2\,[1 - f_2(t)]$$

4. If the eclipse is total, the flux reduces exactly by the luminosity of the obscured star.

In this plot,

- The first deep dip at $t = 1$ day corresponds to the **primary eclipse**.
- The smaller dip at $t = 3$ days corresponds to the **secondary eclipse**.

13.10.5 Radial Velocity Curves

A **binary star system** consists of two stars orbiting their common center of mass. If the orbital plane is inclined relative to the line of sight, the stars will alternately move toward and away from the observer. This motion causes a periodic **Doppler shift** in the spectral lines:

- Motion toward the observer will result in blueshift
- Motion away from the observer will result in redshift

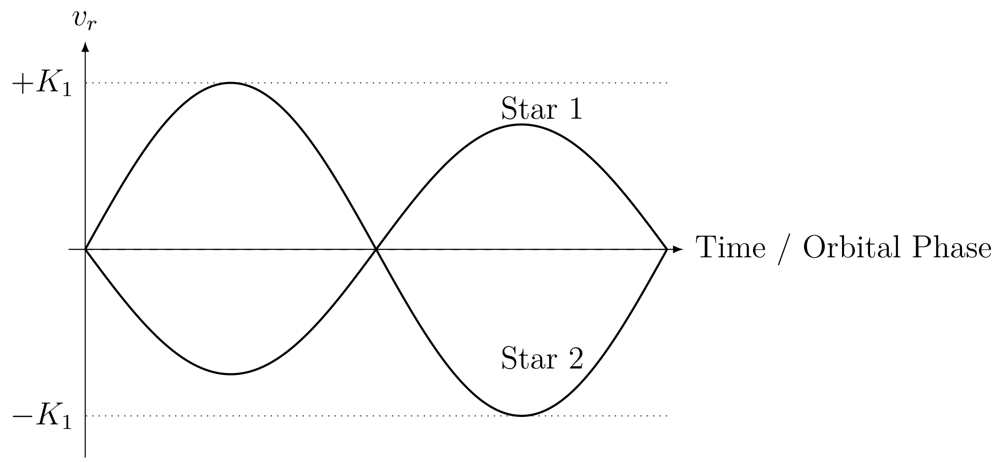
The line-of-sight component of the orbital velocity is called the **radial velocity**. A **radial velocity curve** is a plot of radial velocity v_r versus time t (or orbital phase). It provides direct information about the orbital properties of the binary system.

For a binary star in a circular orbit, the radial velocity varies sinusoidally:

$$v_r(t) = K \sin\left(\frac{2\pi t}{P} + \phi\right)$$

where

- K is the radial velocity amplitude
- P is the orbital period
- ϕ is the phase constant



- The curves are sinusoidal for circular orbits.
- The two stars have opposite phases due to conservation of momentum.
- The more massive star has a smaller velocity amplitude.
- The period of the curve equals the orbital period.

For a binary star in an **elliptical orbit**, the radial velocity of one component is given by:

$$v_r(t) = \gamma + K\left[\cos(\theta(t) + \omega) + e\cos\omega\right]$$

where

- γ is the systemic (center-of-mass) velocity
- K is the radial velocity semi-amplitude
- e is the orbital eccentricity
- ω is the argument of periastron
- $\theta(t)$ is the true anomaly

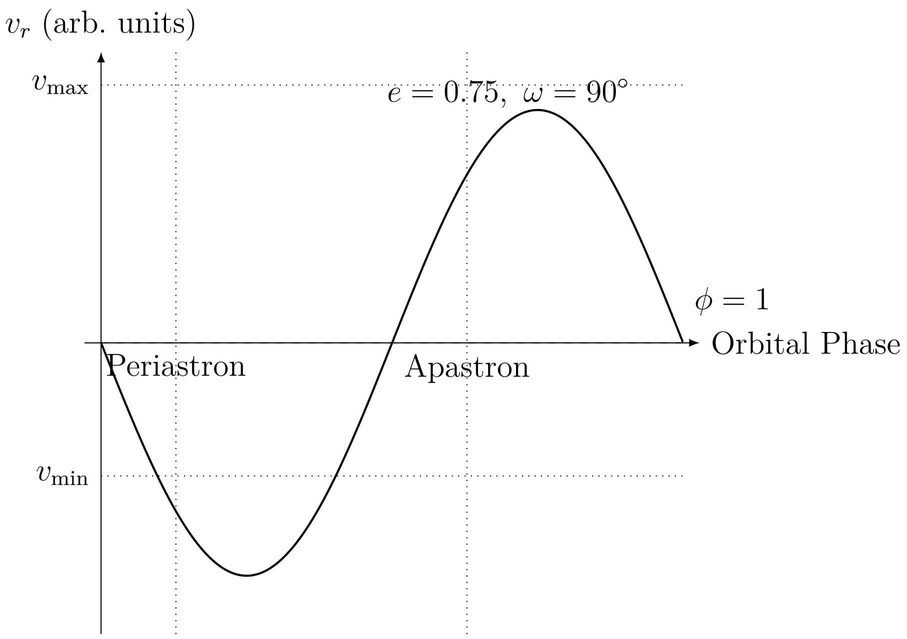
The semi-amplitude K is

$$K = \frac{2\pi a \sin i}{P\sqrt{1-e^2}}$$

where

- a is the semi-major axis of the star's orbit about the barycenter
- i is the inclination angle
- P is the orbital period

The following shows an example of the radial velocity curve:



13.10.6 Roche Lobe

In a close binary star system, the gravitational field experienced by a test particle is determined by the combined gravity of both stars and the centrifugal effect in the rotating frame. The **Roche lobe** of a star is defined as the region around that star within which material is gravitationally bound to it. If a star fills or overflows its Roche lobe, mass transfer to its companion can occur, a key mechanism in interacting binaries such as X-ray binaries, cataclysmic variables, and some exoplanetary systems.

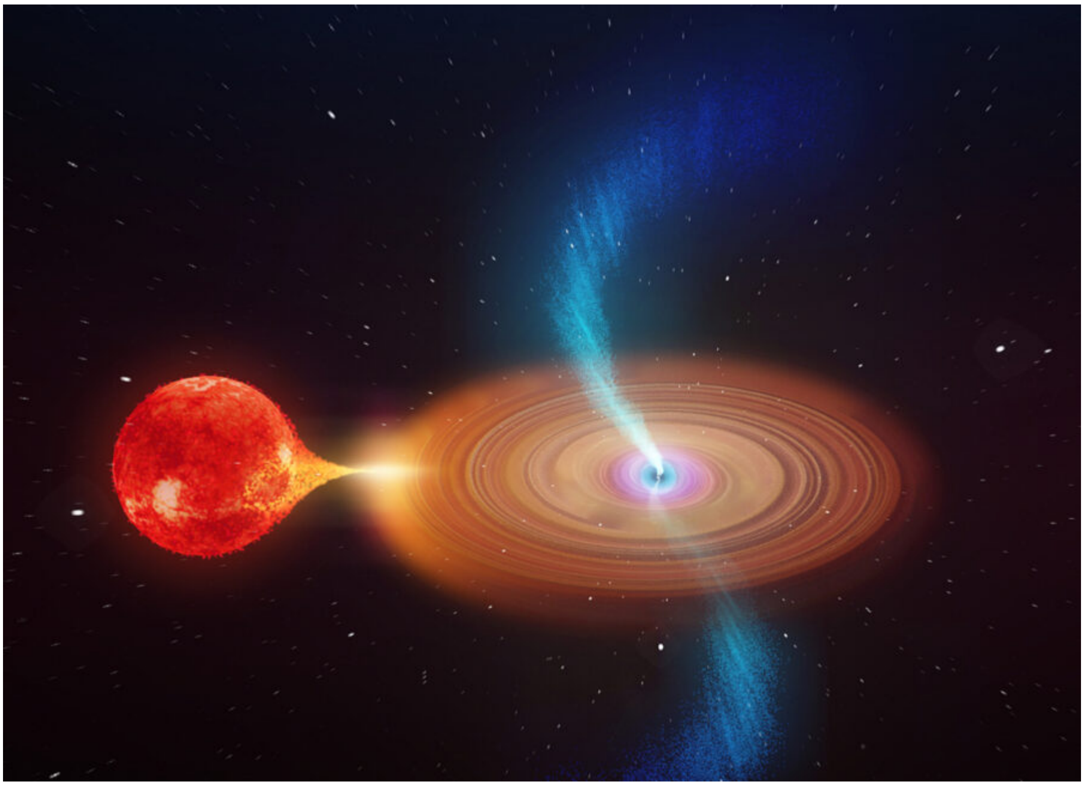


Figure 9: Source: https://root-nation.com/ua/news-ua/it-news-ua/ua-blackhole-warped-accretion-disc/

13.11 Exoplanet

13.11.1 Introduction

An **exoplanet** (or extrasolar planet) is a planet that orbits a star outside our Solar System. The term comes from the Greek "exo" (outside) and "planētēs" (wanderer).

13.11.2 Classes of Exoplanets

- **Hot Jupiters:** Gas giants very close to their stars.
- **Super-Earths:** Planets with masses larger than Earth but smaller than Neptune.
- **Terrestrial planets:** Rocky planets similar to Earth or Mars.
- **Ice giants:** Analogous to Uranus and Neptune.

13.11.3 Spectral Signatures of Possible Life

- Detection of biosignature gases like O_2, O_3, CH_4, and water vapor in exoplanet atmospheres.
- Observed via transmission spectroscopy during planetary transits.

13.11.4 Radial Velocity Method

It is also known as the Doppler method. This technique measures the star's wobble caused by an orbiting planet's gravitational pull.

$$\Delta v = K \left[\frac{P}{2\pi G}\right]^{1/3} \frac{m_p \sin i}{m_s^{2/3}} \frac{1}{\sqrt{1-e^2}}$$

where

- Δv represents the velocity semi-amplitude,
- K is a constant,
- P represents the orbital period,
- m_p represents the planet mass,
- m_s represents the star mass,
- i represents the orbital inclination, and
- e represents the orbital eccentricity.

13.11.5 Transit Method

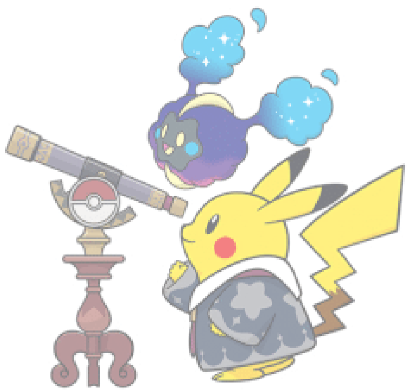
It measures the periodic dimming of a star as a planet passes in front of it.

$$\Delta F = \left(\frac{R_p}{R_s}\right)^2$$

where ΔF is the fractional flux decrease, R_p is planet radius, and R_s is star radius.

13.11.6 Habitable Zone

A habitable zone refers to the region around a star where liquid water could exist on a planet's surface.



14 Cosmology

14.1 Structure of the Universe

14.1.1 Star Clusters

Introduction Star clusters are groups of stars that are gravitationally bound and formed from the same molecular cloud. They provide important insights into stellar evolution and galactic structure. Star clusters are broadly classified into two types:

- **Open Clusters (Galactic Clusters):** These contain a few tens to a few thousand stars. They are loosely bound and typically found in the Galactic disk. Open clusters are relatively young (a few million to a few hundred million years) and often contain hot, massive stars.
- **Globular Clusters:** These are densely packed spherical collections of tens of thousands to millions of stars. They orbit the Galactic halo and are typically very old (10–13 billion years). Globular clusters are rich in low-mass stars and show little gas or dust.

Structurally, a star cluster has a **core** (densest region), a **halo** (more diffuse stars), and in some cases a **tidal radius** where stars may escape due to Galactic gravitational forces.

Luminosity The luminosity L of a star cluster can be calculated by summing the luminosities of all the stars in the cluster:

$$L_{\text{cluster}} = \sum L_{\text{stars}}$$

Where L_{stars} is the luminosity of each star in the cluster.

14.1.2 Galaxies

Introduction Galaxies are vast systems of stars, gas, dust, and dark matter, bound together by gravity. They are the fundamental building blocks of the Universe. Galaxies can be classified based on their structure, composition, and activity:

- **Elliptical galaxies:** Smooth, featureless light distribution, dominated by older stars, little gas or dust.
- **Spiral galaxies:** Flat, rotating disks with spiral arms, containing gas, dust, and young stars.

- **Barred spiral galaxies:** Spiral galaxies with a central bar-shaped structure of stars.
- **Irregular galaxies:** No definite shape, often rich in gas and dust.
- **Active galaxies:** Galaxies with energetic cores (AGN), often emitting strong radiation due to accretion onto supermassive black holes.

Our galaxy is a barred spiral galaxy with several distinct components:

- **Bulge:** Central region, high star density, mostly old stars.
- **Disk:** Contains spiral arms, gas, dust, and young stars.
- **Halo:** Spherical region with globular clusters and dark matter.

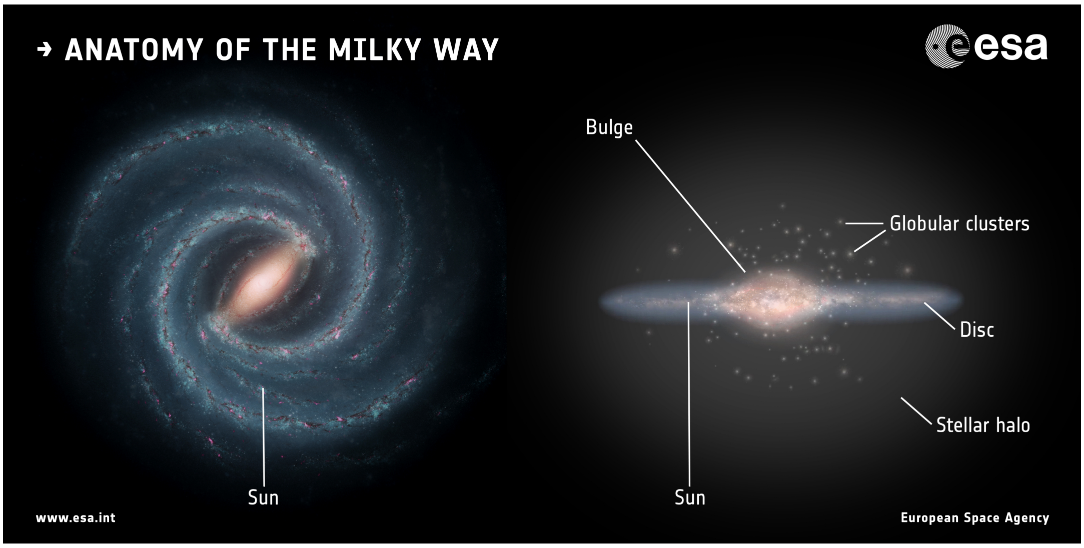
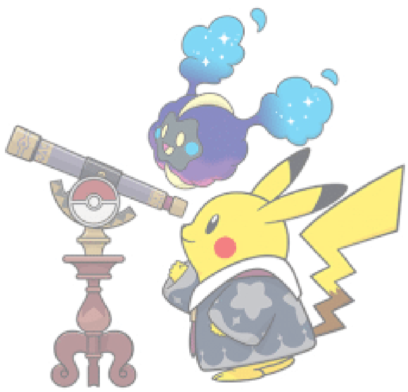


Figure 10: From https://sci.esa.int/web/gaia/-/58206-anatomy-of-the-milky-way

Milky Way The Milky Way is a barred spiral galaxy containing several hundred billion stars, along with interstellar gas, dust, and a dominant dark matter halo. Its stellar disk has a diameter of approximately 30 kpc, while the dark matter halo is believed to extend to radii of order 200 kpc or more. The Milky Way is structured into several main components:

- a thin and thick stellar disk,
- a central bulge and bar,
- a stellar halo,
- an extended dark matter halo.



The Galaxy is not an isolated system. It is surrounded by a population of smaller galaxies gravitationally bound to it, known as **satellite galaxies**, like the Large and Small Magellanic Clouds, and several ultra-faint dwarf galaxies.

Satellite galaxies are typically low-mass systems orbiting the Milky Way within its dark matter halo. They are remnants of the hierarchical assembly process predicted by the ΛCDM cosmological model, in which large galaxies grow through the accretion and merger of smaller ones. Dozens of Milky Way satellites are currently known, and ongoing deep surveys continue to discover new, extremely faint systems.

Galactic Geometry and Coordinates We model the Milky Way as a rotating disk with the Galactic Center (GC) at the origin. Let R be the Galactocentric distance of an object, R_0 be the Galactocentric distance of the Sun, $\Theta(R)$ be the circular rotation speed at radius R, $\Theta_0 = \Theta(R_0)$ be the circular speed of the Sun, l be the Galactic longitude of the object, and b be the Galactic latitude.

Throughout this section we assume $b = 0$ (objects in the Galactic plane), so $\cos b = 1$. The extension to nonzero b is obtained by multiplying the final radial velocity by $\cos b$.

Assume both the Sun and the object move on circular orbits around the Galactic Center. From Galactic geometry, the radial velocity of an object at longitude l is

$$v_r = \left[\Theta(R)\frac{R_0}{R} - \Theta_0\right]\sin l$$

For an object at heliocentric distance d, the Galactocentric radius R is

$$R^2 = R_0^2 + d^2 - 2R_0 d\cos l$$

For longitudes $0^\circ < l < 90^\circ$ or $270^\circ < l < 360^\circ$, the line of sight intersects regions with $R < R_0$. In this case:

- The radial velocity varies monotonically along the line of sight.
- A maximum (or minimum) radial velocity occurs at the **tangent point**.

At the tangent point,

$$R = R_0 \sin l$$

and the radial velocity becomes

$$v_{r,\max} = [\Theta(R_0 \sin l) - \Theta_0 \sin l]$$

For longitudes $90^\circ < l < 270^\circ$, all points along the line of sight satisfy $R > R_0$. In this region,

- There is no tangent point.
- Radial velocity depends on both distance d and the rotation curve.

The general formula

$$v_r = \left[\Theta(R)\frac{R_0}{R} - \Theta_0\right]\sin l$$

still applies, but v_r alone is insufficient to uniquely determine R.

14.2 Large-scale Structure

The **large-scale structure** (LSS) of the Universe refers to the distribution of matter on scales larger than individual galaxies (typically $\gtrsim 1$ Mpc). On these scales, matter is not distributed uniformly, but forms a complex network known as the **cosmic web**:

- **Galaxy clusters**: Dense, gravitationally bound systems of hundreds to thousands of galaxies.
- **Galaxy groups**: Smaller associations of galaxies, often containing a few to tens of members.
- **Filaments**: Elongated structures connecting clusters and groups, containing galaxies and dark matter.
- **Voids**: Large, underdense regions with very few galaxies.
- **Walls / Sheets**: Flattened structures of galaxies separating voids.

Large-scale structure formed from tiny density fluctuations in the early Universe:

- Quantum fluctuations were stretched during cosmic inflation.
- Overdensities grew via **gravitational instability**.
- Dark matter collapsed first, forming gravitational potential wells.
- Baryonic matter later fell into these wells, forming galaxies.

On sufficiently large scales ($\gtrsim 100$ Mpc), the Universe is approximately homogeneous and isotropic, consistent with the **cosmological principle**.

14.3 Cosmological Principle

The **cosmological principle** states that on sufficiently large scales ($\gtrsim 100$ Mpc), the universe is

- **Homogeneous**: Matter and energy are uniformly distributed
- **Isotropic**: No preferred direction in space

14.4 Rotational Curve

In a galaxy, the stars or gas clouds orbit the galactic center due to the gravitational pull of the mass contained within the galaxy. According to Newtonian mechanics, the orbital velocity of an object at a given distance r from the center of a galaxy should behave as

$$v(r) = \sqrt{\frac{GM_{\text{enc}}(r)}{r}}$$

where

- $v(r)$ is the orbital velocity at a distance r from the center and
- $M_{\text{enc}}(r)$ is the enclosed mass within radius r.

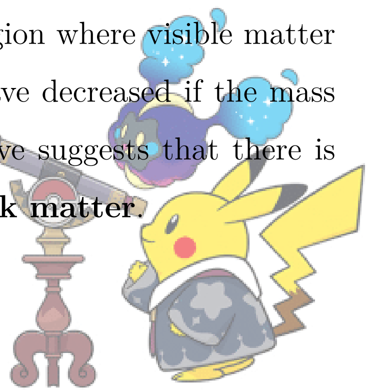
The mass enclosed $M_{\text{enc}}(r)$ depends on the distribution of both visible matter (such as stars and gas) and dark matter within the galaxy.

For a galaxy dominated by visible matter (stars, gas, etc.), the enclosed mass increases with radius, but at larger distances from the center, the mass distribution becomes less dense. Then

$$v(r) \propto \frac{1}{\sqrt{r}}$$

This is called the **Keplerian decline** and is observed in the motion of planets in our solar system. However, observations of galaxies reveal that the rotation curves do not behave this way.

In the 1970s, astronomers like Vera Rubin and Kent Ford observed that the rotation curves of spiral galaxies remain nearly flat at large distances from the center, far beyond the region where visible matter is present. This observation was unexpected, as the rotation velocity should have decreased if the mass distribution followed the visible matter alone. The flatness of the rotation curve suggests that there is additional mass present that is not visible. This mass is what we refer to as **dark matter**.



The flat rotation curves observed at large radii imply that the mass within the galaxy continues to increase even in the outer regions. The orbital velocity $v(r)$ remains constant:

$$v(r) = v_{\text{flat}} \quad \text{for large } r$$

This indicates that the gravitational influence of dark matter is significant at large distances from the galactic center, where visible matter is sparse.

14.5 Hubble's Law

$$v = H_0 \, d$$

where v is the recession velocity, d is the proper distance to the galaxy, and H_0 is the present-day Hubble constant. One can notice that $t_H = \dfrac{1}{H_0}$ is the age of the universe.
Cosmic expansion is described by the **scale factor** $a(t)$, which measures how distances in the Universe change over time. The Hubble parameter is defined using the scale factor:

$$H(t) = \frac{\dot{a}(t)}{a(t)}$$

Light traveling through an expanding Universe also stretches with the expansion. If a photon is emitted at time t_{em} with wavelength λ_{em} and observed today at t_0 with wavelength λ_{obs}, the **cosmological redshift** is defined as

$$1 + z = \frac{\lambda_{\text{obs}}}{\lambda_{\text{em}}}$$

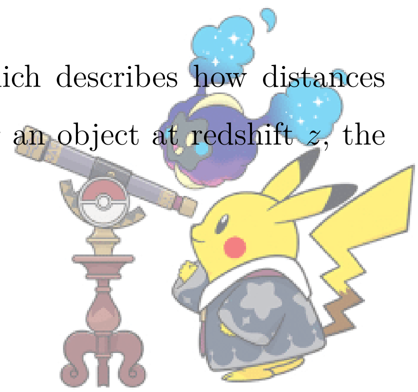
The redshift is directly related to the scale factor:

$$1 + z = \frac{a(t_0)}{a(t_{\text{em}})}$$

14.6 Cosmological Distance Measures

14.6.1 Proper Distance

The proper distance is related to the scale factor $a(t)$ of the universe, which describes how distances between objects in the universe change with time due to the expansion. For an object at redshift z, the



proper distance D_p at a given time is related to the comoving distance D_c by

$$D_\mathrm{p}(t) = a(t) D_\mathrm{c}$$

where

- D_p is the proper distance,
- D_c is the comoving distance,
- $a(t)$ is the scale factor at the time of observation.

14.6.2 Comoving Distance

The **comoving distance** is the distance between two objects as measured in a coordinate system that accounts for the expansion of the universe. Unlike the proper distance, the comoving distance remains constant over time for two objects that are at rest relative to each other in the expanding universe.
The comoving distance D_c at redshift z is related to the scale factor $a(t)$ by

$$D_\mathrm{c} = \int_0^z \frac{c\,dz'}{H(z')}$$

where

- c is the speed of light,
- $H(z')$ is the Hubble parameter as a function of redshift z', and
- z is the redshift at which the object is located.

14.6.3 Luminosity Distance

The **luminosity distance** is the distance to an object based on its observed brightness and its intrinsic luminosity. It is often used for objects like supernovae, which have a known intrinsic luminosity.
The luminosity distance D_L is related to the observed flux f_obs and the intrinsic luminosity L of an object by the inverse square law:

$$f_\mathrm{obs} = \frac{L}{4\pi D_L^2}$$

From this relation, we can solve for the luminosity distance:

$$D_L = \sqrt{\frac{L}{4\pi f_{\text{obs}}}}$$

In a flat universe, the luminosity distance is related to the comoving distance by

$$D_L(z) = (1+z)D_{\text{c}}(z)$$

14.6.4 Angular Diameter Distance

The angular diameter distance is related to the physical size r of an object and its angular size θ (in radians) by the relation

$$\theta = \frac{r}{D_A}$$

14.7 Friedmann Equation

Theorem. 14.1: First Friedmann Equation

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a(t)^2} + \frac{\Lambda c^2}{3}$$

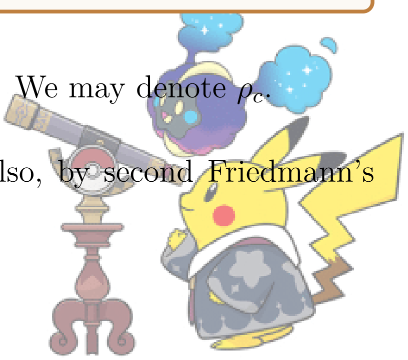
where

- cosmological constant Λ represents homogeneous energy density inherent to empty space (dark energy): $\Lambda = 0$ for flat and static universe (Minkowski universe), $\Lambda > 0$ for expanding universe and $\Lambda < 0$ for contracting universe and
- curvature parameter $k = +1$ for positive curvature, $k = -1$ for negative curvature, $k = 0$ for zero curvature.

Theorem. 14.2: Second Friedmann Equation

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3}\left(\rho - \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}$$

- When $\Lambda = 0$ and $k = 0$, $\rho = \dfrac{3H^2}{8\pi G}$ which is called the important density. We may denote ρ_c.
- Density parameter $\Omega_i := \dfrac{\rho_i}{\rho_c}$. Note that $\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k = 1$. Also, by second Friedmann's



equation and definition,

$$\Omega_m = \frac{8\pi G\rho_m}{3H^2}, \quad \Omega_k = -\frac{kc^2}{a^2H^2}, \quad \Omega_\Lambda = \frac{\Lambda c^2}{8\pi G}$$

- From first and second Friedmann equation,

$$\dot{\rho} + 3H\left(\rho + \frac{p}{c^2}\right) = 0 \text{ (Fluid Equation)}$$

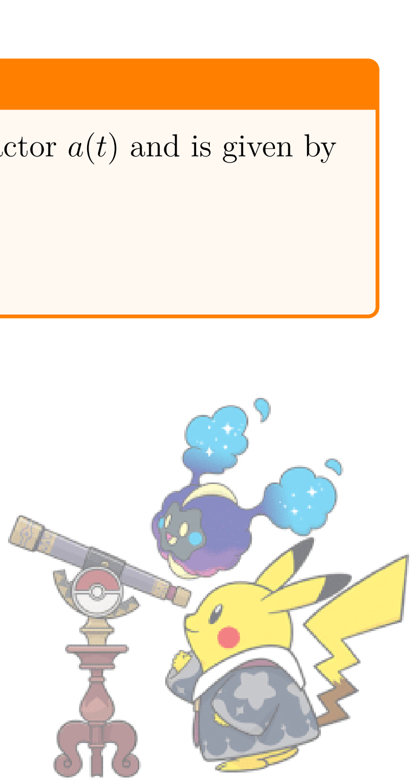
- The ΛCDM model, also known as the Lambda Cold Dark Matter model, is the current standard model of cosmology. It describes the evolution of the Universe from the early hot and dense state to its current large-scale structure. The model is based on the following components:
  - **Cosmological Constant (Λ):** It is responsible for the accelerated expansion of the Universe. Dark energy has an opposing effect to gravity: instead of attracting matter (like gravity does), dark energy exerts a repulsive force. This repulsive force is responsible for causing the accelerated expansion of the Universe, pushing galaxies apart at an ever-increasing rate.
  - **Cold Dark Matter (CDM):** It is also referred to as "cold" because it moves slowly compared to the speed of light. This is in contrast to "hot" dark matter, which would consist of fast-moving particles.

    Cold dark matter is preferred in cosmological models because it allows for the formation of small structures (such as galaxies) early in the Universe's history, which is consistent with observational data.

Definition. 14.1: Hubble parameter

The Hubble parameter $H(t)$ is defined as the rate of change of the scale factor $a(t)$ and is given by

$$H(t) = \frac{\dot{a}(t)}{a(t)}$$



14.8 Equation of State

For different cosmic components,

$$\begin{aligned} \text{Non-relativistic matter (dust):} \quad & p = 0 \quad \Longrightarrow \quad \rho_m \propto a^{-3} \\ \text{Radiation:} \quad & p = \rho c^2/3 \quad \Longrightarrow \quad \rho_r \propto a^{-4} \\ \text{Cosmological constant:} \quad & p = -\rho c^2 \quad \Longrightarrow \quad \rho_\Lambda = \text{constant} \end{aligned}$$

Proof. Rewriting the fluid equation,

$$\frac{d\rho}{dt} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0$$

By chain rule $\dot{\rho} = \dfrac{d\rho}{da}\dot{a}$,

$$\dot{a}\frac{d\rho}{da} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0$$

Assuming $\dot{a} \neq 0$:

$$\frac{d\rho}{da} + \frac{3}{a}\left(\rho + \frac{p}{c^2}\right) = 0$$

By equation of states,

$$p = w\rho c^2$$

where w is a constant for each cosmological component.

$$\frac{d\rho}{da} + \frac{3}{a}(1+w)\rho = 0$$

$$\frac{d\rho}{\rho} = -3(1+w)\frac{da}{a}$$

$$\int_{\rho_i}^{\rho} \frac{d\rho'}{\rho'} = -3(1+w)\int_{a_i}^{a} \frac{da'}{a'}$$

$$\ln\left(\frac{\rho}{\rho_i}\right) = -3(1+w)\ln\left(\frac{a}{a_i}\right)$$

$$\frac{\rho}{\rho_i} = \left(\frac{a}{a_i}\right)^{-3(1+w)}$$

Choosing $a_i = 1$ for present epoch ($\rho_i = \rho_0$):

$$\boxed{\rho(a) = \rho_0 a^{-3(1+w)}}$$

14.9 Big Bang

14.9.1 Singularity

At the very beginning of the universe, all matter and energy were concentrated in a singularity, a point of infinite density and temperature. This singularity is thought to have contained all the space, time, and energy that would later expand to form the universe as we know it.

14.9.2 Cosmic Inflation

Cosmic inflation is a theory that explains the rapid expansion of the universe during the first fractions of a second after the Big Bang. During inflation, the universe expanded exponentially, increasing in size by a factor of at least 10^{26} in a fraction of a second. This theory helps explain several observed features of the universe, such as its large-scale homogeneity and isotropy, and the distribution of galaxies.

14.9.3 Expansion of Space

The Big Bang theory proposes that space itself is expanding. This expansion is not into pre-existing space; rather, it is the stretching of space itself. As space expands, the distances between distant galaxies increase, leading to the observed redshift of light from those galaxies. This expansion of the universe is described mathematically by the **Friedmann equations**.

14.9.4 Phases of the Big Bang

Planck Era (0 to 10^{-43} seconds) The Planck era represents the earliest period of the universe, from time $t = 0$ up to approximately 10^{-43} seconds after the Big Bang. During this time, the universe was incredibly hot and dense, and the fundamental forces (gravity, electromagnetism, the weak nuclear force, and the strong nuclear force) were likely unified in a single force. The exact nature of the physics during this period is unknown, as quantum gravity has yet to be fully understood.

Grand Unification Era (from 10^{-43} to 10^{-36} seconds) During the Grand Unification era, the fundamental forces separated. At the highest energies, the strong, weak, and electromagnetic forces were unified into a single force. As the universe cooled, these forces separated and the strong force, responsible for holding atomic nuclei together, emerged.

Inflationary Era (from 10^{-36} to 10^{-32} seconds) The universe underwent a brief period of exponential expansion during the inflationary era. During this time, the universe expanded by a factor of at least 10^{26} in a fraction of a second. This rapid inflation smoothed out the universe and led to the large-scale homogeneity and isotropy observed today. It also helped set the initial conditions for the formation of the first particles.

Quark Era (from 10^{-12} to 10^{-6} seconds) As the universe continued to cool, quarks, electrons, and other fundamental particles began to form. During the quark era, quarks combined to form protons and neutrons. The temperature and energy were still high enough for these particles to interact and decay frequently.

Hadron Era (from 10^{-6} seconds to 1 second) At around 10^{-6} seconds after the Big Bang, the temperature dropped enough for quarks to combine into hadrons, such as protons and neutrons. This era marked the formation of the first stable atomic nuclei.

Lepton Era (from 1 second to 10 seconds) During the lepton era, the universe was dominated by leptons (such as electrons and neutrinos). These particles were created and annihilated in large quantities. Neutrinos, which were created in abundance, decoupled from the rest of matter at around 10 seconds.

Photon Era (from 10 seconds to 380,000 years) As the universe continued to cool, photons dominated the universe. At this time, matter and radiation were tightly coupled. The universe was opaque because free electrons scattered photons, preventing light from traveling freely. However, the universe continued to expand and cool, and at about 380,000 years after the Big Bang, the universe had cooled enough for atoms to form and photons to travel freely, leading to the decoupling of matter and radiation. This event is known as the **recombination epoch** and is associated with the cosmic microwave background (CMB) radiation, which we observe today.

Recombination and the Formation of Atoms (380,000 years) During recombination, the universe cooled enough for protons and electrons to combine and form neutral hydrogen atoms. This allowed photons to travel freely through space, marking the beginning of the era of **decoupling**. The release of these photons is known as the CMB radiation.

Dark Ages (380,000 years to 1 billion years) After the formation of atoms, the universe entered the "dark ages," a period in which there were no stars or galaxies. During this time, the universe continued

to cool and matter began to clump together due to gravitational attraction. However, it was not until the formation of the first stars and galaxies that the universe became more active.

Reionization (1 billion years to 2 billion years) Reionization occurred when the first stars and galaxies formed and emitted ultraviolet light that reionized the hydrogen gas. This process ended the dark ages and allowed the universe to become transparent to ultraviolet light.

The Modern Universe (Present Day) Since the era of reionization, the universe has continued to expand and evolve. Galaxies, clusters of galaxies, and large-scale structures have formed over billions of years. The observable universe is currently about 93 billion light-years in diameter.

14.10 Cosmic Microwave Background

The Cosmic Microwave Background (CMB) is the oldest light in the universe, originating approximately 380,000 years after the Big Bang. It provides a snapshot of the infant universe when it transitioned from an opaque plasma to a transparent gas.

The CMB originates from the recombination epoch when the universe cooled sufficiently for electrons and protons to combine into neutral hydrogen atoms:

$$p^+ + e^- \longrightarrow \mathrm{H} + \gamma$$

The cosmic microwave background is a faint radiation that fills the universe and is a remnant of the early hot, dense phase of the universe. It was first detected by Penzias and Wilson in 1965 and is often considered the strongest evidence for the Big Bang.

14.11 Gravitational Lensing

14.11.1 Introduction

Gravitational lensing is the bending of light by mass according to general relativity. A mass distribution between a distant source and an observer deflects light rays, producing phenomena such as multiple images, magnification, and distortion of the source.

For a point mass M (in Schwarzschild metric), a light ray with impact parameter b is deflected by an angle

in weak-field. Then, the deflection angle is

$$\hat{\alpha}(b) = \frac{4GM}{bc^2}$$

where G is the gravitational constant and c the speed of light.

We use the thin-lens approximation: the deflection happens effectively at a single lens plane located at angular diameter distance D_L from the observer. Similarly, the source is at distance D_S and the distance between lens and source is D_LS. Let $\boldsymbol{\beta}$ be the true angular position of the source on the sky (unlensed), and $\boldsymbol{\theta}$ the observed angular position of the image. Then

$$\boldsymbol{\beta} = \boldsymbol{\theta} - \boldsymbol{\alpha}(\boldsymbol{\theta})$$

where the scaled deflection angle is

$$\boldsymbol{\alpha}(\boldsymbol{\theta}) = \frac{D_\mathrm{LS}}{D_\mathrm{S}}\,\hat{\boldsymbol{\alpha}}(D_\mathrm{L}\boldsymbol{\theta})$$

For a point mass M located on the optical axis, the scalar form of the lens equation (assuming alignment along a single axis) becomes

$$\beta = \theta - \frac{D_\mathrm{LS}}{D_\mathrm{S}}\frac{4GM}{c^2 D_\mathrm{L}\theta}$$

When the source, lens and observer are perfectly aligned ($\beta = 0$) the image forms a ring (Einstein ring) with angular radius θ_E solving:

$$\theta_\mathrm{E} = \sqrt{\frac{4GM}{c^2}\frac{D_\mathrm{LS}}{D_\mathrm{L}D_\mathrm{S}}}$$

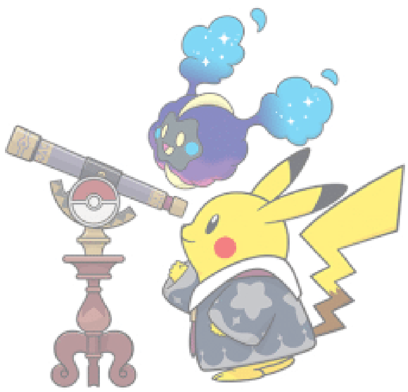
14.11.2 Derivation using Newtonian Mechanics

In Newtonian gravity, the force on a particle of mass m at distance r from M is

$$F = \frac{GMm}{r^2}$$

Only the component perpendicular to the initial direction of motion contributes to the deflection. If the photon moves along the x-axis and passes the mass at distance b, then

$$r = \sqrt{x^2 + b^2}, \quad \sin\theta = \frac{b}{r}$$



The transverse force is therefore

$$F_\perp = F \sin\theta = \frac{GMm}{r^2}\frac{b}{r} = \frac{GMmb}{(x^2+b^2)^{3/2}}$$

The transverse acceleration is

$$a_\perp = \frac{F_\perp}{m} = \frac{GMb}{(x^2+b^2)^{3/2}}$$

The photon moves approximately at constant speed c, so

$$x = ct, \quad dt = \frac{dx}{c}$$

The total change in transverse velocity is

$$\Delta v_\perp = \int_{-\infty}^{\infty} a_\perp \, dt = \frac{1}{c}\int_{-\infty}^{\infty} \frac{GMb}{(x^2+b^2)^{3/2}}\, dx$$

Using the standard integral

$$\int_{-\infty}^{\infty} \frac{b\, dx}{(x^2+b^2)^{3/2}} = \frac{2}{b}$$

we obtain

$$\Delta v_\perp = \frac{2GM}{bc}$$

For small deflections, the bending angle α is approximately the ratio of the transverse velocity change to the speed of light:

$$\alpha \approx \frac{\Delta v_\perp}{c}$$

Hence,

$$\alpha_{\text{Newton}} = \frac{2GM}{bc^2}$$

General Relativity predicts a deflection angle

$$\alpha_{\text{GR}} = \frac{4GM}{bc^2}$$

which is exactly twice the Newtonian result. The additional factor arises from the curvature of space, which is absent in Newtonian gravity.

14.11.3 Derivation using General Relativity

Definition. 14.2: Lagrangian

The **Lagrangian** $\mathcal{L}$ is defined as

$$\mathcal{L}(q, \dot{q}, t) = T(\dot{q}) - V(q)$$

where T is the kinetic energy and V is the potential energy.

Theorem. 14.3: Euler-Lagrange Equation

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{\partial q}$$

Theorem. 14.4: Fermat's Principle

For light propagating in a medium with refractive index $n(\mathbf{r})$, the time taken to travel along a path $\mathcal{C}$ from point A to point B is

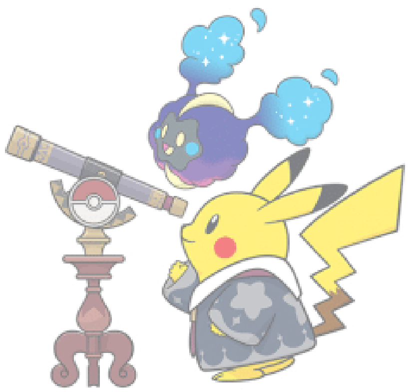
$$T = \int_A^B dt = \int_A^B \frac{ds}{v} = \frac{1}{c}\int_A^B n(\mathbf{r})\, ds$$

where

- $v = c/n$ is the speed of light in the medium
- $ds = \sqrt{dx^2 + dy^2 + dz^2}$ is the infinitesimal path length
- c is the speed of light in vacuum

Fermat's principle states that the actual path $\mathcal{C}$ minimizes (or more generally, makes stationary) the **optical path length**:

$$\boxed{\mathcal{S} = \int_A^B n(\mathbf{r})\, ds}$$



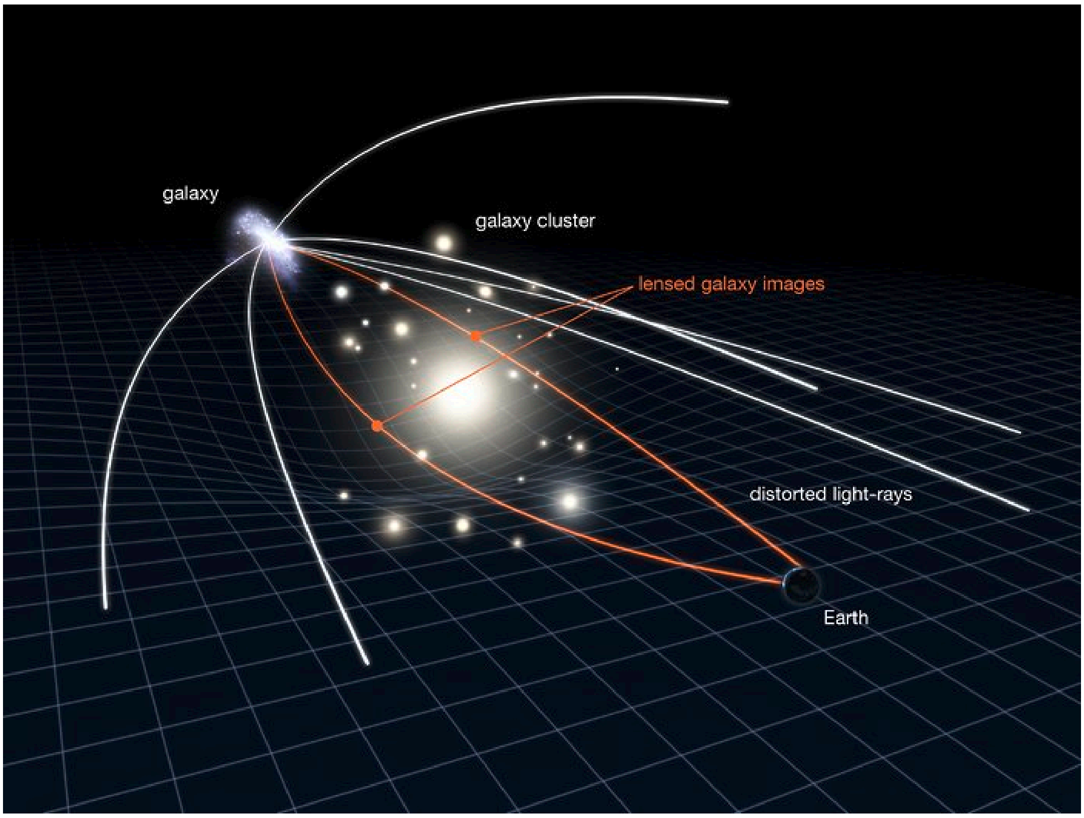


Figure 11: Source: https://esahubble.org/images/heic1106c/

In weak-field approximation of the spacetime metric (which is a situation in which the gravitational field is relatively weak and the spacetime curvature is small),

$$ds^2 = -\left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 + \left(1 - \frac{2\Phi}{c^2}\right) \delta_{ij} dx^i dx^j + \mathcal{O}\left(\frac{\Phi^2}{c^4}\right)$$

where $\Phi = -GM/r$ and $|\Phi| \ll c^2$. Set $c = 1$.

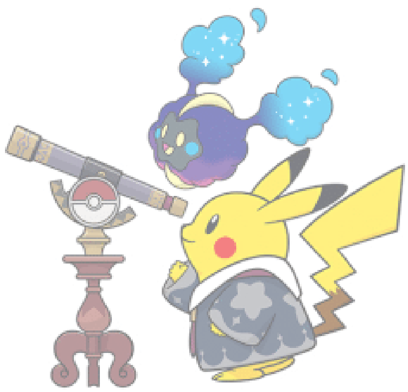
$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)(dx^2 + dy^2 + dz^2) = 0 \implies v \approx 1 + 2\Phi$$

Consider a nearly straight path in the x-y plane with small deflection. Let $y = y(x)$, $z = 0$, with $|y'| \ll 1$. Then

$$dl = \sqrt{1 + (y')^2} dx \approx \left[1 + \frac{1}{2}(y')^2\right] dx$$

The optical path length is

$$S = \int n\, dl \approx \int [1 - 2\Phi(x, y)] \left[1 + \frac{1}{2}(y')^2\right] dx$$
$$\approx \int \left[1 - 2\Phi + \frac{1}{2}(y')^2\right] dx$$



Consider

$$L = \frac{1}{2}(y')^2 - 2\Phi(x, y)$$

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = \frac{\partial L}{\partial y} \implies \frac{d}{dx}(y') = -2\frac{\partial \Phi}{\partial y} \implies \frac{d^2 y}{dx^2} = -2\frac{\partial \Phi}{\partial y}$$

Let $y(x) = b + \varepsilon(x)$ with ε small:

$$\varepsilon''(x) = -2\frac{\partial \Phi}{\partial y}(x, b)$$

The total deflection angle is:

$$\alpha = \Delta y' = \varepsilon'(+\infty) - \varepsilon'(-\infty) = \int_{-\infty}^{\infty} \varepsilon''(x)dx$$

For a point mass $\Phi = -GM/r$ with $r = \sqrt{x^2 + y^2}$:

$$\frac{\partial \Phi}{\partial y} = GM\frac{y}{(x^2 + y^2)^{3/2}}$$

At $y = b$:

$$\alpha = -2GMb\int_{-\infty}^{\infty} \frac{dx}{(x^2 + b^2)^{3/2}}$$

Note that

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + b^2)^{3/2}} = \frac{2}{b^2}$$

Therefore,

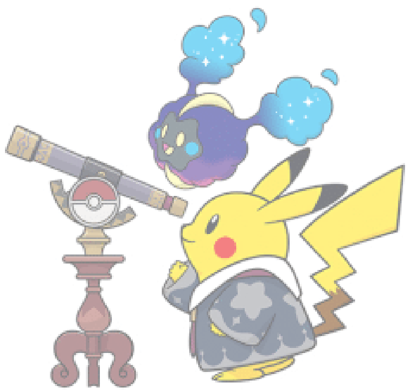
$$\alpha = -2GMb \cdot \frac{2}{b^2} = -\frac{4GM}{b}$$

The magnitude of the deflection (toward the mass) is

$$|\alpha| = \frac{4GM}{b}$$

In the metric, $\Phi \to \Phi/c^2$. Then

$$\boxed{\alpha = \frac{4GM}{bc^2}}$$



14.12 Gravitational Wave

14.12.1 Introduction

Gravitational waves are disturbances in the curvature of spacetime caused by accelerated masses. They were first predicted by Albert Einstein in 1916 as a consequence of his General Relativity theory. Unlike electromagnetic waves, gravitational waves interact weakly with matter, making them challenging to detect but allowing them to carry information about cataclysmic cosmic events.

14.12.2 Chirp Mass

For a binary system with component masses m_1 and m_2, the chirp mass is

$$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}} = \frac{c^3}{G}\left(\frac{5}{96}\pi^{-8/3} f^{-11/3} \dot{f}\right)^{3/5}$$

where f is the orbital frequency.

14.12.3 Binary System

For a binary system with masses m_1 and m_2, and orbital frequency f_{orb}, the gravitational wave luminosity is

$$L_{GW} = \frac{32}{5Gc^5}(2\pi G \mathcal{M} f_{\text{orb}})^{10/3}$$

where r is the orbital separation.

14.13 Accretion Processes

14.13.1 Introduction

Accretion is the process by which matter falls onto a central object, such as a star, black hole, or neutron star, under the influence of gravity.

- Spherical accretion occurs when matter falls radially inward toward a central object in a spherically symmetric manner.
- Disc accretion occurs when matter, due to its angular momentum, forms a rotating disc as it falls toward a central object.

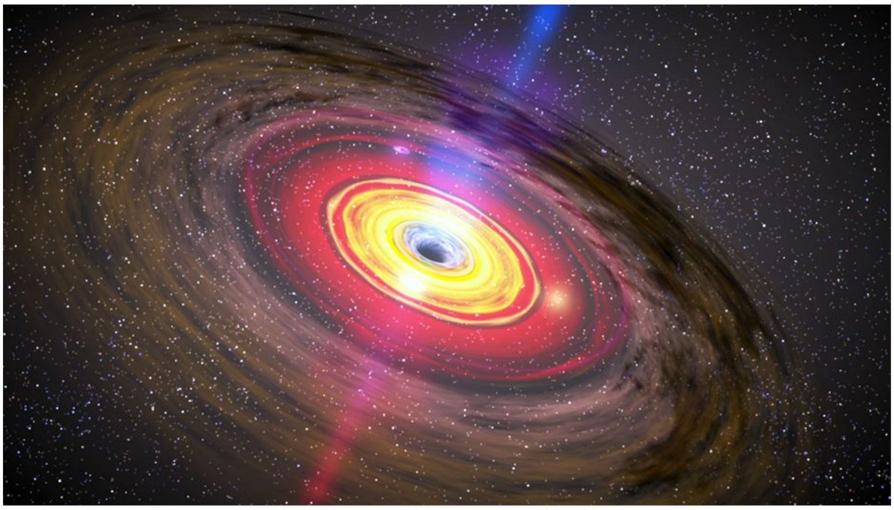


Figure 12: Source: https://www.skyatnightmagazine.com/space-science/accretion-disk

14.13.2 Eddington Luminosity

Consider a spherical surface with radius r centered around a light source, where the total luminosity of the source is L. The energy flux (the energy per unit time passing through a unit area) at a distance r from the source is given by the total luminosity divided by the surface area of a sphere with radius r. The surface area of a sphere is

$$A = 4\pi r^2$$

Hence, the energy flux at distance r is:

$$F = \frac{L}{4\pi r^2}$$

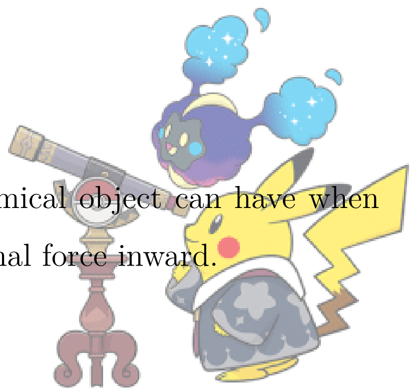
Now, consider the nature of radiation pressure. Photons carry momentum, and when they strike a surface, they transfer momentum to it. The radiation pressure is related to the energy flux by the relationship:

$$E = pc \implies P_{\text{rad}} = \frac{1}{A}\frac{dp}{dt} = \frac{F}{c}$$

where c is the speed of light, and this formula assumes that the radiation is isotropic and that the photon momentum transfer is fully efficient in transferring momentum to the surface. Then

$$P_{\text{rad}} = \frac{1}{c} \cdot \frac{L}{4\pi r^2} = \frac{L}{4\pi r^2 c}$$

The **Eddington luminosity**, L_{Edd}, is the maximum luminosity an astronomical object can have when there is a balance between the radiation pressure outward and the gravitational force inward.



The radiation pressure on an object is given by

$$P_{\text{rad}} = \frac{L}{4\pi r^2 c}$$

where L is the luminosity, r is the radius of the object, and c is the speed of light. The gravitational force is given by

$$F_{\text{grav}} = \frac{GMm}{r^2}$$

where M is the mass of the central object, m is the mass of the accreting material, and G is the gravitational constant. When balanced,

$$\frac{L}{4\pi r^2 c} = \frac{GMm}{r^2}$$

Simplifying this equation:

$$L = \frac{4\pi GMmc}{r^2}$$

For the Eddington luminosity, we consider the maximum luminosity for the material to remain bound to the central object without being blown away by radiation pressure. Using the fact that the material consists of hydrogen, for which the mass of an electron is m_e and the Thomson scattering cross-section is σ_T, we can calculate the Eddington luminosity as follows:

$$L_{\text{Edd}} = \frac{4\pi GMm_e c}{\sigma_T}$$

where $\sigma_T \approx 6.65 \times 10^{-25}\,\text{m}^2$ is the effective area that quantifies the likelihood of an electron scattering a photon through Thomson scattering.

14.14 Cosmic Distance Ladder

14.14.1 Introduction

The cosmic distance ladder is a succession of methods by which astronomers determine the distances to celestial objects. Each rung of the ladder provides information that allows calibration of the next method, enabling measurements from the Solar System to the edge of the observable universe.

The Cosmic Distance Ladder

RADAR & Parallax

0.001–1000 parsecs

Direct geometric methods

↓ Calibrates

Standard Candles: Cepheids & RR Lyrae

1–30 Mpc

Pulsating variable stars with period-luminosity relationship

↓ Calibrates

Tully-Fisher & Faber-Jackson Relations

10–200 Mpc

Galaxy luminosity correlated with rotation speed/velocity dispersion

↓ Calibrates

Type Ia Supernovae

100–1000 Mpc

"Standardizable" candles from white dwarf explosions

↓ Calibrates

Hubble's Law

$>$100 Mpc

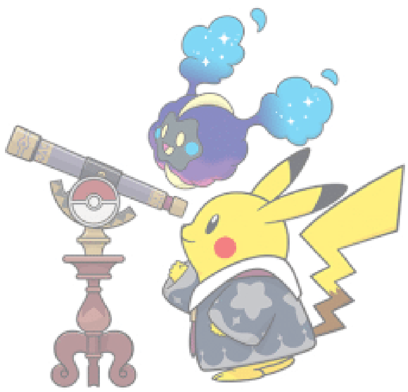
Cosmological redshift-distance relation

Figure 13: The hierarchy of distance measurement techniques in astronomy. Each rung calibrates the next.

14.14.2 Radar Ranging

For objects within our Solar System, we can use radar to measure distances directly:

- Transmit radio waves toward a planet or asteroid
- Measure time delay for echo to return: Δt
- Distance: $d = \frac{c \cdot \Delta t}{2}$ where c is the speed of light
- Limited to $\sim$10 AU (within Solar System)



14.14.3 Stellar Parallax

Parallax uses Earth's orbit as a baseline to measure distances to nearby stars:

$$d(\text{parsecs}) = \frac{1}{\theta(\text{arcseconds})}$$

where

- d = distance in parsecs (1 pc $\approx$ 3.26 light-years)
- θ = parallax angle in arcseconds
- Baseline = 1 AU (astronomical unit)

14.14.4 Standard Candles: Cepheid Variables

Stellar Variability Stellar variability is classified into two broad categories:

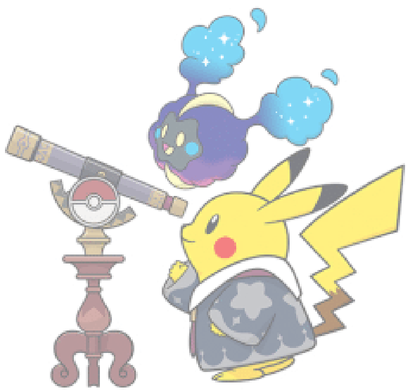
1. **Regular Variability:**
    - **Pulsating Stars:** These stars, such as Cepheid variables, exhibit periodic changes in brightness due to expansions and contractions of their outer layers.
    - **Eclipsing Binaries:** These systems consist of two stars orbiting each other, and their light curves vary periodically due to one star eclipsing the other.
2. **Irregular Variability:**
    - **Flare Stars:** These stars exhibit sudden, unpredictable increases in brightness due to magnetic activity.
    - **Cataclysmic Variables:** These stars experience large variations in brightness due to mass transfer in binary systems.

Cepheid Variables Cepheid variables are pulsating stars whose period correlates with luminosity:

$$M = a \cdot \log_{10}(P) + b$$

where

- M is the absolute magnitude (intrinsic brightness),



- P is the pulsation period in days, and
- a, b are the calibration constants.

Once the absolute magnitude M is known from the period, the distance modulus formula gives the distance.

14.14.5 Faber-Jackson Relations

For elliptical galaxies, the velocity dispersion correlates with luminosity:

$$L \propto \sigma^{\beta}$$

where

- σ is the stellar velocity dispersion
- $\beta \approx 4$ empirically

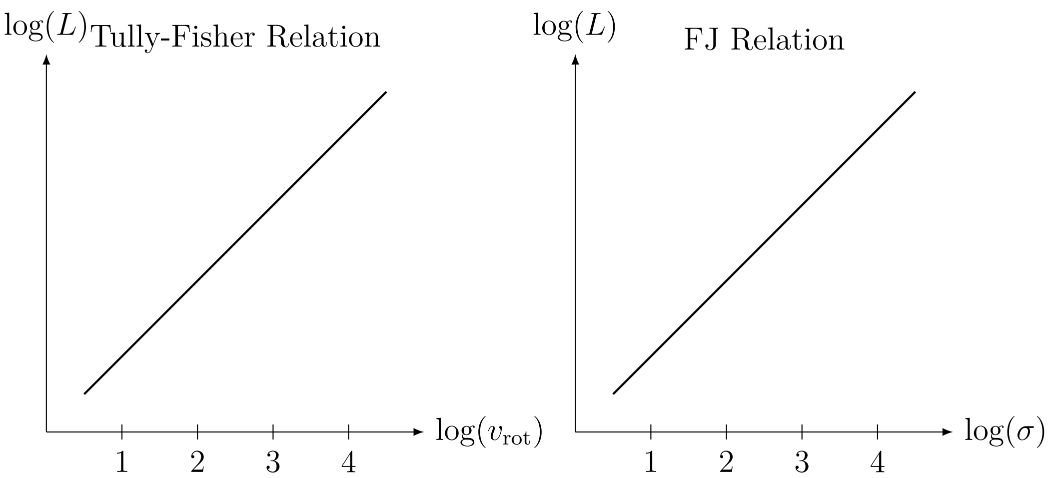


Figure 14: The Tully-Fisher and Faber-Jackson relations allow estimation of galaxy distances from measurable kinematic properties.

14.14.6 Type Ia Supernovae

Type Ia supernovae are extremely luminous and serve as excellent "standardizable candles":

- Result from thermonuclear explosion of white dwarf reaching Chandrasekhar limit ($\sim 1.4\ M_{\odot}$)
- Peak luminosity: $\sim 10^{10}\ L_{\odot}$ (as bright as entire galaxy)
- Can be observed at cosmological distances

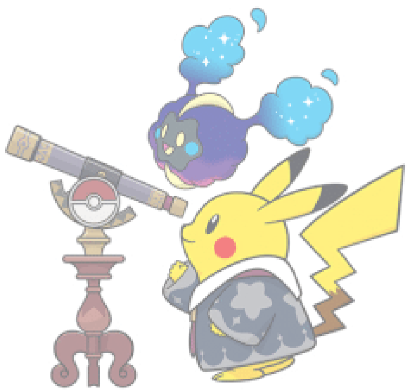
- Light curve shape correlates with peak luminosity (Phillips relationship)

The distance is calculated from

$$\mu = m_B - M_B + \alpha(s-1) - \beta c$$

where

- μ is the distance modulus
- m_B is the apparent magnitude in B-band
- M_B is the absolute magnitude (calibrated)
- s is the light curve shape parameter
- c is the color correction
- α, β are the calibration coefficients



15 Interstellar Medium

15.1 Introduction

The **interstellar medium** refers to the matter that exists in the space between stars within a galaxy. It is composed of gas, dust, and cosmic rays.
It can be categorized into several distinct phases based on the temperature and density of the material:

- **Neutral Gas**: Consists of neutral hydrogen (H) and molecular hydrogen (H_2), often found in molecular clouds.
- **Ionized Gas**: Composed of ionized hydrogen (H^+) and other ionized elements, typically found in regions such as HII regions.
- **Dust**: Microscopic solid particles that can range in size from nanometers to microns, contributing to the absorption and scattering of light.
- **Cosmic Rays**: High-energy particles, primarily protons and atomic nuclei, that travel through the ISM.

Some important regions within the interstellar medium include:

- **HII Regions**: These are regions of ionized hydrogen, created by the ultraviolet radiation from young, hot stars. They are often observed in emission lines such as Hα.
- **Molecular Clouds**: These are cold, dense regions of the interstellar medium where molecules such as H_2 are found. They are often the birthplaces of stars.
- **Warm Ionized Medium**: A diffuse component of the interstellar medium, where the gas is partially ionized and has a temperature around 10^4 K.

15.2 Fluid Dynamics

15.2.1 Stress Tensor

In continuum mechanics, the **stress tensor $\boldsymbol{\sigma}$** is a fundamental concept that describes the internal forces acting within a material or fluid.

For a general three-dimensional space, the stress tensor $\boldsymbol{\sigma}$ is a 3×3 matrix, with components σ_{ij} where i and j refer to the directions (or axes) in space:

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

where

- $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ are the normal stress components in the x, y, and z directions.
- $\sigma_{xy}, \sigma_{xz}, \sigma_{yz}$ are the shear stress components.
- The matrix is symmetric: $\sigma_{ij} = \sigma_{ji}$.

15.2.2 Tensor Product

Definition. 15.1: Tensor Product

Let V and W be vector spaces over a field $\mathbb{F}$. The **tensor product** $V \otimes W$ is a vector space equipped with a bilinear map

$$\otimes : V \times W \longrightarrow V \otimes W$$

satisfying the following universal property: For every bilinear map $\phi : V \times W \longrightarrow U$ to any vector space U, there exists a unique linear map $\tilde{\phi} : V \otimes W \longrightarrow U$ such that the following diagram commutes:

$$\begin{array}{ccc} V \times W & \xrightarrow{\otimes} & V \otimes W \\ & \searrow^{\phi} & \downarrow \tilde{\phi} \\ & & U \end{array}$$

Tensor product of two vectors $\mathbf{A}$ and $\mathbf{B}$ is given by

$$(\mathbf{A} \otimes \mathbf{B})_{ij} = A_i B_j$$

15.2.3 Divergence Theorem

Theorem. 15.1: Divergence Theorem

$$\int_{\partial V} \mathbf{F} \cdot \mathbf{n} \, dS = \int_V \nabla \cdot \mathbf{F} \, dV$$

where

- $\mathbf{F}$ is a vector field,
- V is the volume enclosed by the surface ∂V,
- $\mathbf{n}$ is the outward-pointing unit normal vector on the surface ∂V, and
- dS is the surface area element on ∂V

15.2.4 Continuity Equation and Momentum Equation

Let $\rho(\mathbf{x}, t)$ denote the mass density and $\mathbf{v}(\mathbf{x}, t)$ the velocity field of a fluid. The total mass within a fixed control volume V with boundary ∂V and outward unit normal $\mathbf{n}$ is

$$M_V(t) = \int_V \rho \, dV$$

The principle of mass conservation states that the rate of change of mass within V equals the negative of the net mass flux through the boundary:

$$\frac{d}{dt} \int_V \rho \, dV = -\int_{\partial V} \rho \, \mathbf{v} \cdot \mathbf{n} \, dS$$

As V is fixed in space,

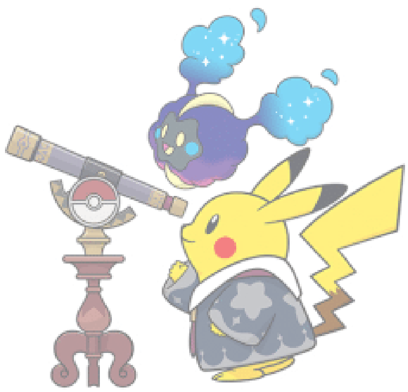
$$\int_V \frac{\partial \rho}{\partial t} \, dV = -\int_{\partial V} \rho \, \mathbf{v} \cdot \mathbf{n} \, dS$$

By the divergence theorem,

$$\int_V \frac{\partial \rho}{\partial t} \, dV = -\int_V \nabla \cdot (\rho \mathbf{v}) \, dV$$

Hence,

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right) dV = 0$$



As V is arbitrary, the integrand must vanish everywhere and

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0}$$

which is the local form of continuity equation.

By the material derivative $D/Dt = \partial/\partial t + \mathbf{v} \cdot \nabla$,

$$\boxed{\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0}$$

Consider the momentum of fluid within a fixed control volume V:

$$\int_V \rho \mathbf{v}\, dV$$

$$\frac{d}{dt}\int_V \rho \mathbf{v}\, dV = -\int_{\partial V} \rho \mathbf{v}(\mathbf{v} \cdot \mathbf{n})\, dS + \int_{\partial V} \boldsymbol{\sigma} \cdot \mathbf{n}\, dS + \int_V \rho \mathbf{f}\, dV$$

where

- $\boldsymbol{\sigma}$ is the stress tensor,
- $\mathbf{f}$ is the body force per unit mass (e.g. gravity).

Using the divergence theorem,

$$\int_{\partial V} \rho \mathbf{v}(\mathbf{v} \cdot \mathbf{n})\, dS = \int_V \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v})\, dV, \quad \int_{\partial V} \boldsymbol{\sigma} \cdot \mathbf{n}\, dS = \int_V \nabla \cdot \boldsymbol{\sigma}\, dV$$

Therefore,

$$\int_V \left(\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) - \nabla \cdot \boldsymbol{\sigma} - \rho \mathbf{f} \right) dV = 0$$

As V is arbitrary,

$$\boxed{\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f}}$$

which is the **Cauchy momentum equation in conservation form**. Using the continuity equation, it can be rewritten in material-derivative form:

$$\boxed{\rho \frac{D\mathbf{v}}{Dt} = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f}}$$

15.2.5 Euler's Equation (Special Case of Navier-Stokes Equation)

From

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f}$$

where ρ is the density of the fluid, $\mathbf{v}$ is the velocity of the fluid, $\boldsymbol{\sigma}$ is the stress tensor, and $\mathbf{f}$ represents external body forces per unit mass.

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) = \rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \frac{\partial \rho}{\partial t}$$

$$\nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) = \mathbf{v} \cdot \nabla(\rho \mathbf{v}) = \mathbf{v} \cdot \nabla(\rho)\mathbf{v} + \rho \mathbf{v} \cdot \nabla \mathbf{v}$$

For an inviscid fluid (no viscosity), the stress tensor only contains the pressure term:

$$\boldsymbol{\sigma} = -p\mathbf{I}$$

where p is the pressure and $\mathbf{I}$ is the identity matrix. The divergence of the stress tensor is

$$\nabla \cdot \boldsymbol{\sigma} = -\nabla p$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho \mathbf{v} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \rho \mathbf{f}$$

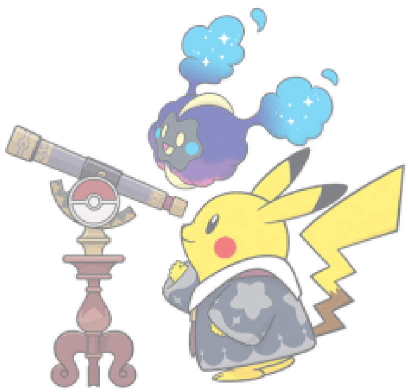
For an incompressible flow (constant ρ):

$$\frac{\partial \rho}{\partial t} = 0 \quad \text{and} \quad \nabla \rho = 0$$

This implies that the terms $\mathbf{v}\frac{\partial \rho}{\partial t}$ and $\mathbf{v} \cdot \nabla \rho \mathbf{v}$ vanish. Hence,

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \rho \mathbf{f}$$

$$\boxed{\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p + \mathbf{f}}$$



15.3 Case Study

15.3.1 Background

The interstellar medium consists of gas and dust at various densities and temperatures, organized into molecular clouds, atomic gas, and ionized phases. Gravitational collapse occurs when self-gravity overcomes pressure support, leading to star formation. The interstellar medium exhibits a wide range of densities:

Phase	**Density, ρ (g cm^{-3})**	**Collapse Timescale, $t_{\rm ff}$**
Molecular Cloud (diffuse)	4.2×10^{-22}	$\sim 10^7$ yr
Molecular Cloud (dense core)	4.2×10^{-20}	$\sim 10^6$ yr
Atomic HI Cloud	4.2×10^{-24}	$\sim 10^8$ yr
HII Region	4.2×10^{-25}	$\sim 10^9$ yr

15.3.2 Derivation

- Continuity equation (mass conservation):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

- Euler equation (momentum conservation):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p - \nabla \Phi$$

- Poisson equation for the gravitational potential where $\nabla^2 f = \nabla \cdot (\nabla f)$ and $\nabla^2 \Phi = f$:

$$\nabla^2 \Phi = 4\pi G \rho$$

Consider an uniform background:

$$\rho = \rho_0 + \delta\rho, \quad \mathbf{v} = \delta\mathbf{v}, \quad p = p_0 + \delta p, \quad \Phi = \Phi_0 + \delta\Phi$$

To isolate pure gravitational collapse, we consider the pressure-less limit:

- Linearized continuity equation:

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot \delta\mathbf{v} = 0$$

- Linearized Euler equation (no pressure):

$$\frac{\partial \delta\mathbf{v}}{\partial t} = -\nabla \delta\Phi$$

- Linearized Poisson equation:

$$\nabla^2 \delta\Phi = 4\pi G\, \delta\rho$$

Therefore,

$$\frac{\partial}{\partial t}(\nabla \cdot \delta\mathbf{v}) = -\nabla^2 \delta\Phi$$

and then

$$\frac{\partial}{\partial t}(\nabla \cdot \delta\mathbf{v}) = -4\pi G\, \delta\rho$$

Also,

$$\frac{\partial^2 \delta\rho}{\partial t^2} + \rho_0 \frac{\partial}{\partial t}(\nabla \cdot \delta\mathbf{v}) = 0$$

Finally,

$$\frac{\partial^2 \delta\rho}{\partial t^2} + \rho_0 \left[-4\pi G\, \delta\rho\right] = 0$$

Hence,

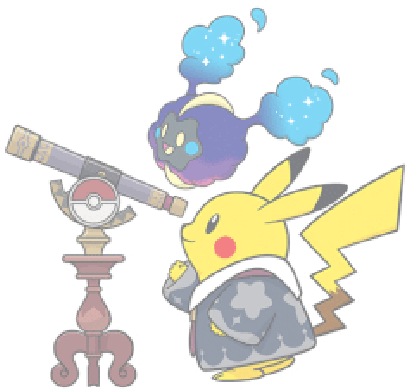
$$\boxed{\frac{\partial^2 \delta\rho}{\partial t^2} = 4\pi G \rho_0\, \delta\rho}$$

which is in the form of $\ddot{\delta\rho} = \omega^2 \delta\rho$ with

$$\omega^2 \equiv 4\pi G \rho_0 > 0$$

The $e^{\omega t}$ mode represents exponential growth of density perturbations which represent gravitational collapse. The $e^{-\omega t}$ mode decays and is typically not physically relevant for collapse initial conditions. The **e-folding time** (characteristic growth/collapse timescale) is

$$\tau \equiv \frac{1}{\omega} = \frac{1}{\sqrt{4\pi G \rho_0}} \sim \frac{1}{\sqrt{G \rho_0}}$$



16 Study of the Earth

16.1 Tides

Tides are the periodic rise and fall of sea levels caused by the gravitational forces exerted by the Moon and the Sun on Earth.

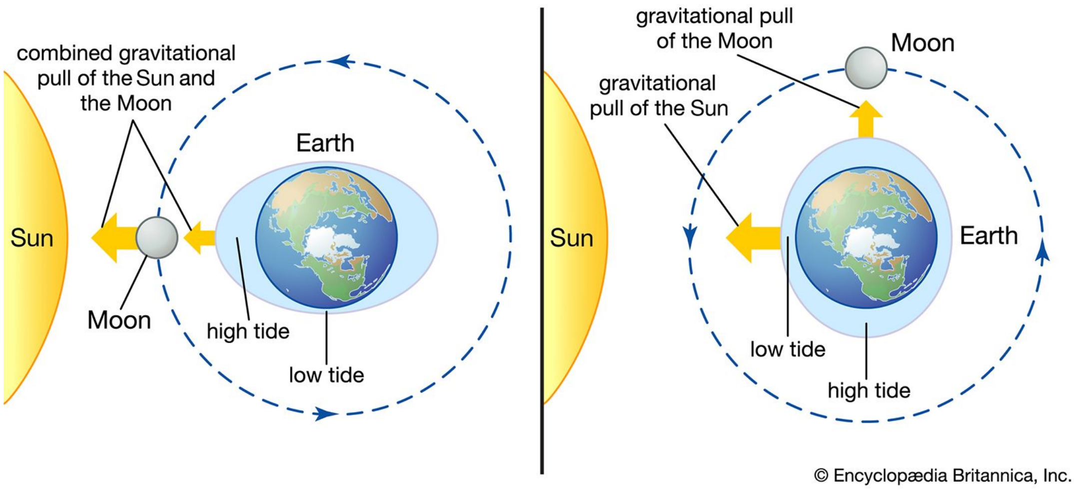
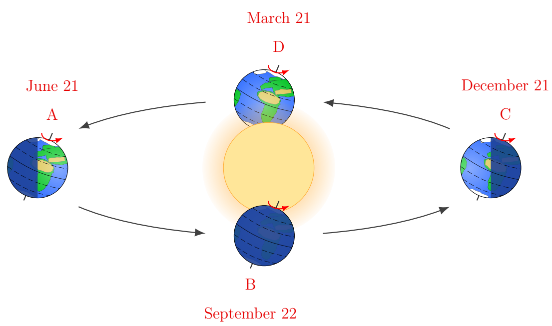


Figure 15: Source: https://www.britannica.com/science/tide

16.2 Seasons

Seasons are caused by the tilt of Earth's axis (23.5°) relative to its orbit around the Sun.

- Summer occurs in the hemisphere tilted toward the Sun.
- Winter occurs in the hemisphere tilted away from the Sun.
- Spring and Autumn occur when neither hemisphere is tilted toward the Sun.



16.3 Factors Influencing Climate

Climate depends on several natural factors:

- Latitude: Distance from the equator affects temperature.
- Altitude: Higher altitudes are cooler.
- Ocean Currents: Warm or cold currents can affect coastal climates.
- Topography: Mountains can block winds and affect rainfall.
- Human Activities: Urbanization and greenhouse gases influence climate.

16.4 Eclipses

16.4.1 Solar Eclipse

It occurs when the Moon comes between the Earth and Sun, blocking sunlight.

16.4.2 Lunar Eclipse

It occurs when the Earth comes between the Sun and Moon, casting a shadow on the Moon.

16.5 Space Weather

16.5.1 Introduction

Space weather refers to the dynamic conditions in Earth's space environment, primarily influenced by the Sun.

16.5.2 Solar Wind

The solar wind is a stream of charged particles released from the upper atmosphere of the Sun, called the corona. It affects the Earth's magnetosphere and can disrupt satellite communications.

16.5.3 Solar Flares

Solar flares are sudden bursts of radiation from the Sun. They release energy across the electromagnetic spectrum, which can impact radio communications and GPS systems on Earth.

16.5.4 Coronal Mass Ejections (CMEs)

CMEs are massive bursts of solar wind and magnetic fields rising above the solar corona. They can trigger geomagnetic storms that affect satellites, power grids, and auroras.

16.5.5 Aurorae

Aurorae (Northern and Southern Lights) are caused by charged particles from the Sun interacting with Earth's magnetic field and atmosphere.

16.6 Meteor Showers

Meteor showers occur when Earth passes through the debris left by a comet.

16.7 Equinoxes

An **equinox** occurs twice a year, when the Sun crosses the celestial equator. On these dates, day and night are approximately equal in length at all latitudes. The two equinoxes are:

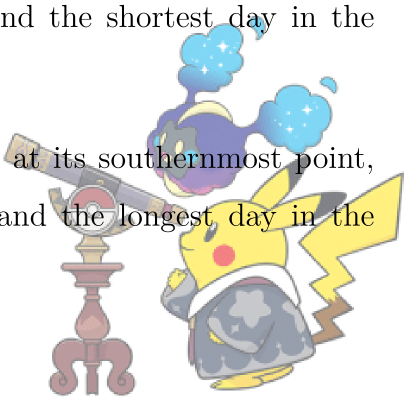
- **Vernal Equinox**: Occurs around March 20th or 21st, marking the start of spring in the northern hemisphere.
- **Autumnal Equinox**: Occurs around September 22nd or 23rd, marking the start of autumn in the northern hemisphere.

At the equinoxes, the Sun rises directly in the east and sets directly in the west.

16.8 Solstices

A **solstice** occurs twice a year when the Sun reaches its highest or lowest point in the sky at noon, relative to the celestial equator. This results in the longest and shortest days of the year.

- **Summer Solstice**: Occurs around June 21st or 22nd. The Sun is at its northernmost point, resulting in the longest day of the year in the northern hemisphere and the shortest day in the southern hemisphere.
- **Winter Solstice**: Occurs around December 21st or 22nd. The Sun is at its southernmost point, resulting in the shortest day of the year in the northern hemisphere and the longest day in the southern hemisphere.



16.9 Solar Declination

Solar declination is the angle between the rays of the Sun and the plane of the Earth's equator. It varies throughout the year, reaching $+23.5°$ during the summer solstice and $-23.5°$ during the winter solstice. At the equinoxes, the solar declination is $0°$, meaning the Sun is directly above the equator.

17 Study of the Moon

17.1 Precession

Precession is the slow, conical motion of the Earth's rotation axis caused primarily by the gravitational torque of the Sun and Moon on Earth's equatorial bulge.

$$\text{Rate of precession} \approx 50.3'' \text{ per year}$$

Proof. Consider the Earth as an oblate spheroid with equatorial radius R_e and polar radius R_p. The Earth's equatorial bulge experiences a gravitational torque due to the Sun (or Moon):

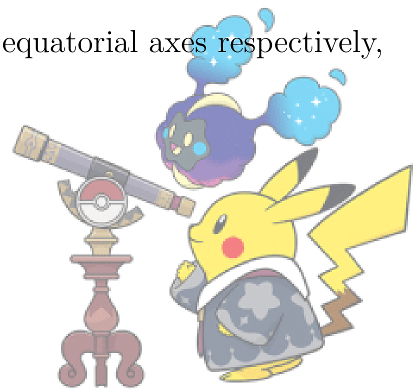
$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

The magnitude of the torque is proportional to the Earth's moment of inertia difference and the gravitational force:

$$\tau \approx \frac{3GM_s}{2r^3}(C - A)\sin 2\theta$$

where

- G is the gravitational constant,
- M_s is the mass of the Sun,
- r is the Earth-Sun distance,
- C and A are the Earth's principal moments of inertia about polar and equatorial axes respectively, and
- θ is the obliquity of the ecliptic ($\approx 23.5°$).



The precessional angular velocity Ω_p is given by the ratio of the torque to the Earth's spin angular momentum $L = C\omega$:

$$\Omega_p = \frac{\tau}{C\omega} = \frac{3GM_s}{2r^3}\frac{C - A}{C\omega}\cos\theta$$

Here, ω is the Earth's spin angular velocity. The $\cos\theta$ factor appears due to the component of torque perpendicular to the spin axis. For an oblate Earth,

$$C - A = \frac{2}{5}M_e R_e^2 J_2$$

where

- M_e is the Earth's mass,
- $J_2 \approx 1.0826 \times 10^{-3}$ is the dynamical flattening coefficient.

The precessional angular velocity becomes

$$\Omega_p = \frac{3GM_s}{2r^3} \cdot \frac{\frac{2}{5}M_e R_e^2 J_2}{C\omega}\cos\theta$$

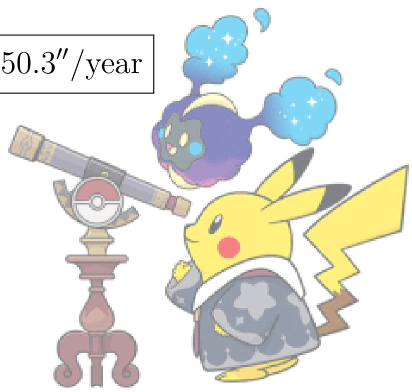
Using $C \approx \frac{2}{5}M_e R_e^2$, we simplify:

$$\Omega_p \approx \frac{3GM_s}{2r^3\omega}J_2\cos\theta$$

Substitute the values

$$\begin{aligned} G &= 6.674 \times 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}} \\ M_s &= 1.989 \times 10^{30}\,\mathrm{kg} \\ r &= 1.496 \times 10^{11}\,\mathrm{m} \\ \omega &= 7.292 \times 10^{-5}\,\mathrm{rad/s} \\ J_2 &= 1.0826 \times 10^{-3} \\ \theta &= 23.5^\circ \end{aligned}$$

$$\boxed{\text{Rate of precession} = \Omega_p \times 206264.8'' \times 3.156 \times 10^7\,\mathrm{s/year} \approx 50.3''/\mathrm{year}}$$



17.2 Nutation

Nutation refers to small periodic oscillations superimposed on the precessional motion. These are caused by the varying positions of the Moon and Sun relative to Earth, leading to deviations in the tilt of Earth's axis.

17.3 Libration

Libration is the apparent oscillation of the Moon that allows observers on Earth to see slightly more than half of its surface over time. There are three main types:

- **Longitudinal libration:** Due to the eccentricity of the Moon's orbit.
- **Latitudinal libration:** Due to the tilt of the Moon's axis relative to its orbital plane.
- **Diurnal libration:** Due to the rotation of the Earth and the observer’ s changing viewpoint.

18 Study of the Solar System

18.1 Formation

The Solar System formed about 4.6 billion years ago from a giant molecular cloud composed of gas and dust. The process can be divided into several stages.

18.1.1 Nebular Hypothesis

The nebular hypothesis explains that the Solar System formed from a rotating disk of gas and dust. This cloud, known as the solar nebula, collapsed under its own gravity, leading to the formation of the Sun at its center and the planets from the remaining material. The key stages in the formation of the Solar System are as follows:P

1. **Collapse of the Solar Nebula**: The gas and dust cloud began to contract due to gravity. As it contracted, it started to rotate faster, forming a flat, rotating disk.
2. **Formation of the Sun**: At the center of the disk, the temperature and pressure increased, leading to nuclear fusion, which ignited the Sun.

3. **Accretion of Planets**: In the outer regions of the disk, dust and gas began to clump together to form planetesimals, which further collided and merged to form planets, moons, and other small bodies.

4. **Clearing the Nebula**: The young Sun's solar wind cleared away the remaining gas and dust, leaving behind the current structure of the Solar System.

18.1.2 Differentiation and Evolution

After the initial formation, the Solar System underwent several evolutionary processes:

- **Differentiation**: The early planets were molten, and heavier materials sank toward their cores while lighter materials rose to the surface.

- **Late Heavy Bombardment**: During the early stages, the planets were frequently bombarded by leftover planetesimals, causing cratering on their surfaces.

- **Orbital Evolution**: Gravitational interactions between planets and other bodies in the Solar System led to changes in their orbits over time.

18.2 Structure and Components of the Solar System

The Solar System is composed of the Sun and all the objects that are bound by its gravitational field, including planets, moons, asteroids, comets, and other small bodies.

18.2.1 The Sun

The Sun is the central star of the Solar System, providing the gravitational force that holds the system together. It is composed primarily of hydrogen and helium and accounts for approximately 99.86% of the total mass of the Solar System. The Sun' s core is where nuclear fusion occurs, generating the energy that powers the Sun and supports life on Earth.

18.2.2 Planets

The modern scientific definition of a **planet** was formally established by the International Astronomical Union (IAU) in 2006. According to the IAU, a celestial body is classified as a planet if it satisfies **all** of the following three criteria:

- **It orbits the Sun.** The object must revolve around the Sun, distinguishing planets of the Solar System from moons, which orbit planets, and from extrasolar (exoplanetary) systems.
- **It has sufficient mass for its self-gravity to overcome rigid body forces, so that it assumes hydrostatic equilibrium (a nearly round shape).** This condition ensures that the object is massive enough for gravity to shape it into a roughly spherical form.
- **It has cleared the neighbourhood around its orbit.** The object must be gravitationally dominant in its orbital region, meaning it has either accreted or scattered most other bodies of comparable size near its orbit.

An object that meets the first two criteria but has not cleared its orbital neighbourhood is classified as a **dwarf planet** (e.g. Pluto, Ceres, and Eris). There are eight planets in the Solar System, divided into two main categories:

Terrestrial Planets The terrestrial planets are rocky bodies with solid surfaces and relatively high densities. They are located in the inner Solar System:

- **Mercury**
- **Venus**
- **Earth**
- **Mars**

These planets are characterized by thin or moderate atmospheres, slow rotation rates compared to gas giants, and a composition dominated by silicate rocks and metals.

Gas Giants and Ice Giants The giant planets are large, massive planets with thick atmospheres and no well-defined solid surfaces. They are further divided into gas giants and ice giants:

- **Gas giants**: Jupiter, Saturn
- **Ice giants**: Uranus, Neptune

Giant planets possess strong gravitational fields, extensive systems of moons, and prominent ring systems.

18.2.3 Smaller Bodies

The Solar System also contains numerous smaller bodies, including

- **Asteroids**: Rocky bodies that primarily orbit between Mars and Jupiter in the asteroid belt.
- **Comets**: Icy bodies that often have highly elliptical orbits, and develop tails when they approach the Sun.
- **Meteoroids**: Small fragments of asteroids or comets that can enter Earth's atmosphere and cause meteor showers.

18.2.4 Kuiper Belt and Oort Cloud

The outer regions of the Solar System are populated by icy bodies and dwarf planets:

- **Kuiper Belt**: A region beyond Neptune that contains icy bodies, including dwarf planets like Pluto.
- **Oort Cloud**: A hypothetical cloud of icy bodies that is believed to surround the Solar System at great distances, thought to be the source of long-period comets.

19 Study of the Sun

19.1 Composition

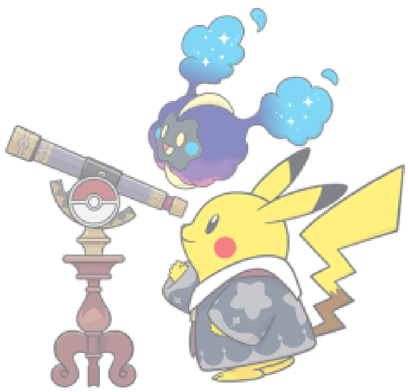
The Sun is primarily composed of hydrogen and helium, with trace amounts of heavier elements. Hydrogen nuclei (protons) undergo nuclear fusion in the solar core:

$$4p \longrightarrow {}^{4}\text{He} + 2e^{+} + 2\nu_e + \gamma + 26.7 \text{ MeV}$$

powering the Sun through the proton-proton chain.

19.2 Internal Structure

1. **Core**:
    - Radius: $\sim 0.2\ R_{\odot}$
    - Site of nuclear fusion: $4p \longrightarrow {}^{4}\text{He} + 2e^{+} + 2\nu_e + 26.7$ MeV



- Temperature: $\sim 1.5 \times 10^7$ K

2. **Radiative Zone**:

    - Energy transported by radiation
    - Temperature gradient decreases outward

3. **Convective Zone**:

    - Energy transported by convection
    - Outer $\sim 30\%$ of Sun’s radius
    - Convection cells cause granulation on the photosphere

19.3 Atmosphere

- **Photosphere**: Visible surface; $T \sim 5800$ K
- **Chromosphere**: Above photosphere; hotter than photosphere; $T \sim 10^4$ K
- **Corona**: Outermost layer; $T \sim 10^6$ K; source of solar wind

19.4 Solar Surface Activities

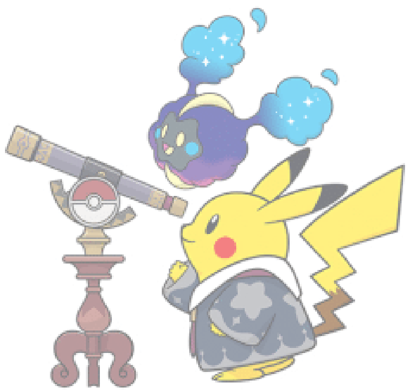
19.4.1 Sunspots

Sunspots are temporary dark regions on the solar photosphere (visible surface of the Sun) caused by strong magnetic fields.

19.4.2 Solar Wind

The solar wind is a continuous outflow of plasma from the solar corona. The solar wind consists mainly of

- Protons (p^+)
- Electrons (e^-)
- Alpha particles ($\alpha = {}^4\text{He}^{2+}$)



The solar corona has high temperature ($T \sim 1 \times 10^6$ K). The sound speed is

$$c_s \sim \sqrt{\frac{k_B T}{m_p}}$$

As the corona cannot remain static with such high thermal pressure, the plasma expands outward, forming the solar wind. Plasma is often called the fourth state of matter. It is a fully or partially ionized gas, meaning that a significant fraction of the atoms or molecules are electrically charged (ions and electrons). The **heliosphere** is the region of space dominated by the solar wind and the Sun's magnetic field, extending well beyond the orbit of Pluto.
The **magnetosphere** is the region around a planet where the planetary magnetic field dominates the motion of charged particles, protecting the planet from the solar wind.

19.5 Magnetohydrodynamics

Magnetohydrodynamics (MHD) is the physical theory that describes the dynamics of electrically conducting fluids in the presence of magnetic fields. Such fluids include plasmas, liquid metals, and saltwater. MHD combines principles from:

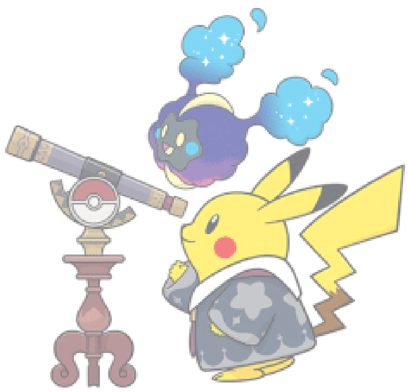
- **Fluid dynamics** (Navier–Stokes equations),
- **Electromagnetism** (Maxwell's equations), and
- **Thermodynamics**

In the MHD approximation, the plasma is treated as a single conducting fluid rather than as separate ions and electrons. This approximation is valid when the characteristic length scales are much larger than the particle mean free paths and the Debye length. The basic set of ideal MHD equations consists of:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

where ρ is the mass density and $\mathbf{v}$ is the fluid velocity.

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mathbf{J} \times \mathbf{B} + \rho \mathbf{g}$$



where p is the gas pressure, $\mathbf{B}$ is the magnetic field, $\mathbf{J}$ is the current density, and $\mathbf{g}$ is the gravitational acceleration.

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B})$$

where η is the magnetic diffusivity. In **ideal MHD**, $\eta = 0$.

$$\nabla \cdot \mathbf{B} = 0$$

which expresses the absence of magnetic monopoles. The Sun is composed primarily of ionized hydrogen and helium, making it an excellent example of a natural MHD system. Different layers of the Sun exhibit different MHD behaviors:

- **Solar interior**: Dense plasma with strong coupling between flow and magnetic fields.
- **Photosphere and chromosphere**: Visible surface where magnetic structures emerge.
- **Corona**: Extremely hot, low-density plasma dominated by magnetic forces.

The Sun possesses a large-scale magnetic field generated by a **solar dynamo**. This dynamo operates in the convection zone and is driven by

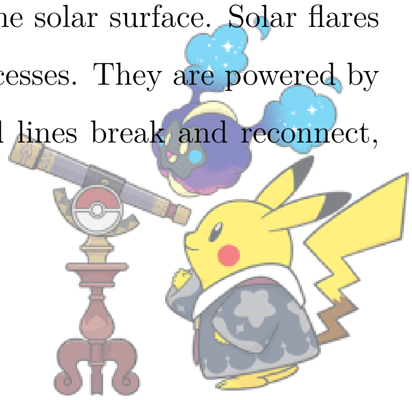
- Differential rotation,
- Turbulent convective motions,
- Plasma conductivity.

MHD equations describe how plasma flows stretch, twist, and amplify magnetic field lines, converting kinetic energy into magnetic energy. Sunspots are regions of strong magnetic fields that inhibit convective heat transport. In MHD terms, the magnetic pressure

$$p_{\mathrm{mag}} = \frac{B^2}{2\mu_0}$$

partially balances the gas pressure, leading to cooler and darker regions on the solar surface. Solar flares and coronal mass ejections (CMEs) are dramatic manifestations of MHD processes. They are powered by **magnetic reconnection**, a non-ideal MHD process in which magnetic field lines break and reconnect, rapidly releasing stored magnetic energy. This energy is converted into

- Thermal heating,



- Particle acceleration,
- Electromagnetic radiation.

MHD predicts the existence of wave modes in magnetized plasmas. One important example is the **Alfvén wave**, which propagates along magnetic field lines with speed

$$v_A = \frac{B}{\sqrt{\mu_0 \rho}}$$

Proof. Consider an ideal, perfectly conducting plasma with uniform background magnetic field $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$, uniform mass density ρ, no background flow ($\mathbf{v}_0 = 0$), and small perturbations in velocity and magnetic field. The governing equations are

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \mathbf{J} \times \mathbf{B}$$

For incompressible, transverse perturbations, pressure gradients can be neglected, giving

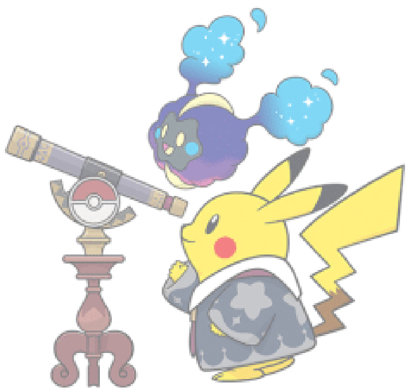
$$\rho \frac{\partial \mathbf{v}}{\partial t} = \mathbf{J} \times \mathbf{B}$$

By Ampère's law neglecting displacement current,

$$\mathbf{J} = \frac{1}{\mu_0} \nabla \times \mathbf{B}$$

Note that

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$



By Faraday's law,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

By Gauss's law,

$$\nabla \cdot \mathbf{B} = 0$$

In magnetohydrodynamics, the generalized Ohm's law is

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J}$$

where η is the resistivity. For an **ideal** conductor ($\eta = 0$), this reduces to

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B}$$

Substitute $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$ into Faraday's law:

$$\nabla \times (-\mathbf{v} \times \mathbf{B}) = -\frac{\partial \mathbf{B}}{\partial t}$$

Rearranging gives

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

Let the magnetic field and velocity be written as

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{b}, \qquad \mathbf{v} = \mathbf{v}_1$$

where $\mathbf{b}$ and $\mathbf{v}_1$ are small perturbations. To first order, the equation of motion becomes

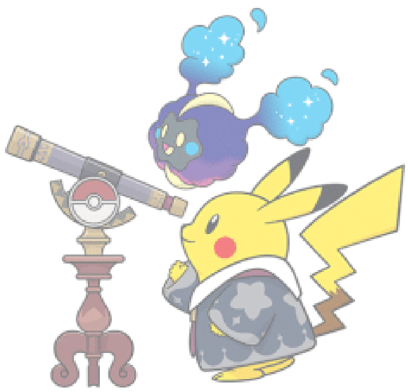
$$\rho \frac{\partial \mathbf{v}_1}{\partial t} = \frac{1}{\mu_0} (\nabla \times \mathbf{b}) \times \mathbf{B}_0$$

The linearized induction equation is

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0)$$

Take the time derivative of the equation of motion:

$$\rho \frac{\partial^2 \mathbf{v}_1}{\partial t^2} = \frac{1}{\mu_0} \left(\nabla \times \frac{\partial \mathbf{b}}{\partial t} \right) \times \mathbf{B}_0$$



Substitute the induction equation:

$$\rho \frac{\partial^2 \mathbf{v}_1}{\partial t^2} = \frac{1}{\mu_0} \left[\nabla \times (\nabla \times (\mathbf{v}_1 \times \mathbf{B}_0)) \right] \times \mathbf{B}_0$$

For transverse waves propagating along $\mathbf{B}_0$ (take $\partial/\partial z \neq 0$ only), this simplifies to

$$\frac{\partial^2 \mathbf{v}_1}{\partial t^2} = \frac{B_0^2}{\mu_0 \rho} \frac{\partial^2 \mathbf{v}_1}{\partial z^2}$$

The above equation is a standard wave equation of the form

$$\frac{\partial^2 \mathbf{v}_1}{\partial t^2} = v_A^2 \frac{\partial^2 \mathbf{v}_1}{\partial z^2}$$

from which we identify the Alfvén speed as

$$\boxed{v_A = \frac{B_0}{\sqrt{\mu_0 \rho}}}$$

20 Human and Robotic Exploration within the Solar System

20.1 Human Exploration of the Solar System

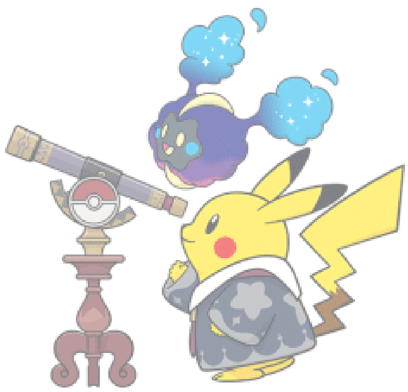
Human exploration of the Solar System involves sending astronauts beyond Earth to explore, study, and potentially inhabit other celestial bodies.

- **Purpose**: advancing scientific knowledge, developing space technology, inspiring societies, and ensuring long-term survival of humanity.
- **Major challenges**: long-duration exposure to microgravity, cosmic radiation, limited medical support, psychological isolation, and safe re-entry.
- **Key destinations**: the Moon for testing technologies, Mars for long-term exploration, and near-Earth asteroids for scientific and resource studies.
- **Approach**: step-by-step expansion using space stations, lunar missions, and sustainable life-support systems.

20.2 Planetary Missions

Planetary missions are robotic or crewed missions designed to explore planets, moons, asteroids, and comets.

- **Flyby missions**: provide brief but valuable observations with minimal fuel and mission complexity.
- **Orbiter missions**: allow long-term monitoring, global mapping, and atmospheric studies.
- **Landers and rovers**: enable direct surface analysis, geology, and chemical investigations, but require complex entry, descent, and landing systems.
- **Scientific value**: reveal planetary formation history, climate evolution, and potential habitability.



Part II: Data Analysis

21 Probability

21.1 Introduction

The probability of an event A is a number between 0 and 1 that represents the likelihood of the event occurring. It is defined as

$$P(A) = \frac{\text{Number of favorable outcomes}}{\text{Total number of possible outcomes}}$$

For example, the probability of getting heads in a fair coin toss is:

$$P(\text{Heads}) = \frac{1}{2}$$

The probability of the complement of an event A, denoted as A^c, is:

$$P(A^c) = 1 - P(A)$$

For example, if $P(\text{Heads}) = \frac{1}{2}$, then the probability of getting tails, $P(\text{Tails}) = \frac{1}{2}$.
The probability of event A occurring given that event B has occurred is called conditional probability and is denoted as $P(A|B)$. It is given by

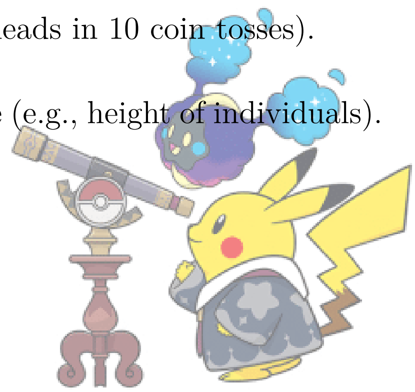
$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

This is the probability of A given B, assuming $P(B) > 0$.

21.2 Random Variables

A random variable is a numerical outcome of a random phenomenon. It can be classified as either discrete or continuous:

- **Discrete Random Variable:** Takes distinct values (e.g., number of heads in 10 coin tosses).
- **Continuous Random Variable:** Takes any value within a given range (e.g., height of individuals).



The probability mass function (PMF) gives the probability of each possible outcome for discrete random variables. For example, the PMF of a fair die roll (with outcomes $1, 2, 3, 4, 5, 6$) is

$$P(X = x) = \frac{1}{6}, \quad x \in \{1, 2, 3, 4, 5, 6\}$$

The probability density function (PDF) is used for continuous random variables. The probability that a continuous random variable X takes a value in the interval $[a, b]$ is given by the integral of the PDF over that interval:

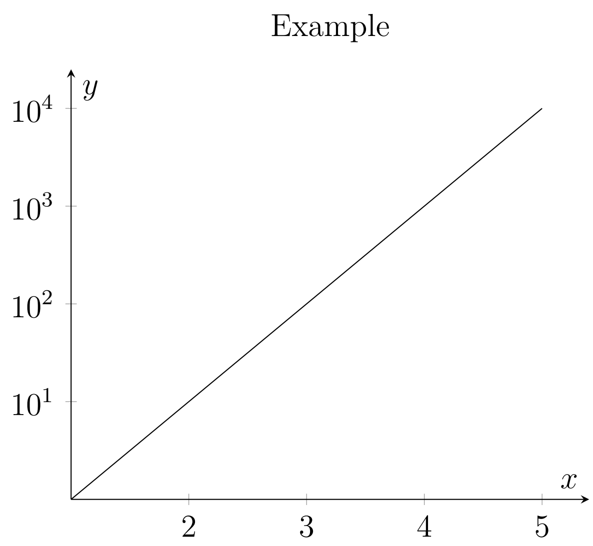
$$P(a \leq X \leq b) = \int_a^b f_X(x)\, dx$$

where $f_X(x)$ is the PDF of X.

22 Linear and Logarithmic Scale

In a linear scale, the distance between points is the same for each increment. The axis values increase by a constant amount.

In a logarithmic scale, the distance between values is proportional to the logarithm of the values:



23 Measure of Central Tendancy

Definition. 23.1: Mean

The mean $\bar{x}$ is the sum of all values divided by the number of observations. For a sample of size n,

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Definition. 23.2: Median

The median is the middle value when the data is ordered from least to greatest.

- If n is odd: The median is the value at position $\frac{n+1}{2}$.
- If n is even: The median is the average of the two middle values.

Definition. 23.3: Mode

The mode is the value that appears most frequently in the dataset.

24 Measure of Dispersion

24.1 Basic Concepts

Definition. 24.1: Quartile

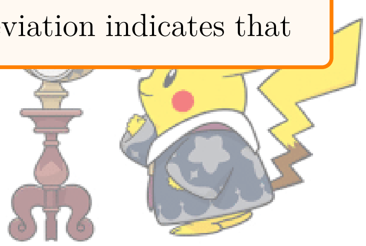
Quartiles are statistical measures that divide an ordered dataset into four equal parts. They help describe the spread and distribution of the data. The three quartiles are:

$$Q_1 \quad \text{(first quartile)}, \qquad Q_2 \quad \text{(second quartile or median)}, \qquad Q_3 \quad \text{(third quartile)}$$

These quartiles divide the data into four segments, each containing 25% of the observations.

Definition. 24.2: Standard Deviation

Standard deviation is a measure of the spread or dispersion of a set of numerical data. It tells us how much the values deviate, on average, from the mean (average). A small standard deviation indicates that the data points are clustered close to the mean, while a large standard deviation indicates that



the data are spread out. For the data points

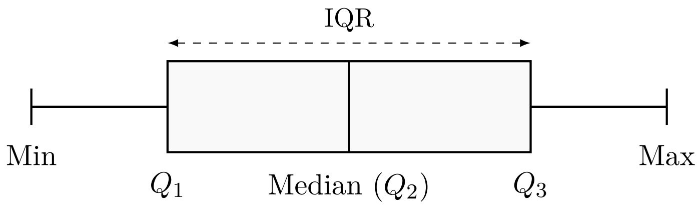
$$x_1, x_2, x_3, \ldots, x_N$$

the standard deviation is defined as

$$\boxed{\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2}}$$

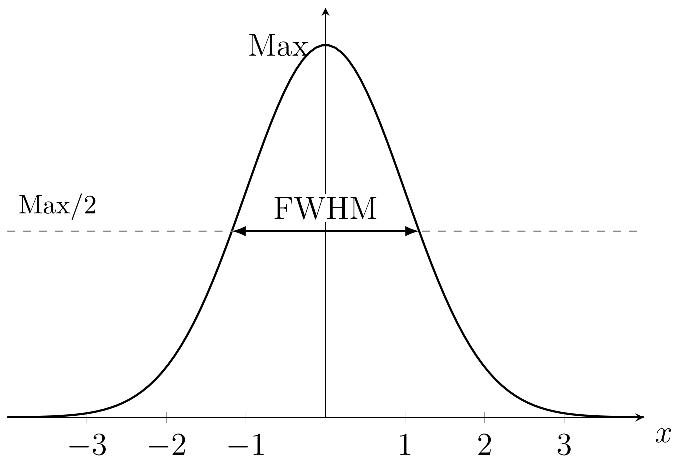
24.2 Box Plots (Box-and-Whisker)

A box plot is a graphical method for displaying the distribution of numerical data using five key summary values:



25 Full Width at Half Maximum

FWHM is the width of a curve measured at half of its maximum amplitude.



Example. For Gaussian function

$$f(x) = Ae^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where

- A is the amplitude (maximum value) of the Gaussian,
- μ is the mean (center) of the distribution,
- σ is the standard deviation (width) of the distribution.

From

$$Ae^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{A}{2}$$

$$e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{2}$$

$$-\frac{(x-\mu)^2}{2\sigma^2} = \ln\frac{1}{2} = -\ln 2$$

$$(x-\mu)^2 = 2\sigma^2 \ln 2$$

The points x_1 and x_2 are

$$x = \mu \pm \sigma\sqrt{2\ln 2}$$

Therefore, the FWHM is

$$\text{FWHM} = x_2 - x_1 = 2\sigma\sqrt{2\ln 2}$$

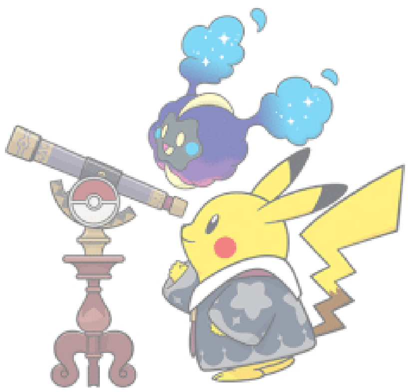
26 Error Analysis

In experimental measurements, quantities have uncertainties or errors. If a function depends on multiple variables:

$$z = f(x, y, \dots)$$

and $x, y, \dots$ have uncertainties $\Delta x, \Delta y, \dots$, then the approximate uncertainty in z is

$$\boxed{\Delta z \approx \sqrt{\left(\frac{\partial z}{\partial x}\Delta x\right)^2 + \left(\frac{\partial z}{\partial y}\Delta y\right)^2 + \dots}}$$



27 Regression Analysis

27.1 Linear regression

27.1.1 Introduction

Linear regression is a fundamental supervised learning algorithm used to model the relationship between a **scalar response** (dependent variable, Y) and **one or more explanatory variables** (independent variables, X). For **simple linear regression** (one independent variable), the relationship is modeled by a straight line:

$$\hat{y} = \beta_0 + \beta_1 x$$

Where $\hat{y}$ is the predicted response, x is the independent variable, β_0 is the intercept, and β_1 is the slope.

27.1.2 Least Squares Method

The goal is to find the optimal coefficients (β_0, β_1) that make the line best fit the observed data points (x_i, y_i). The best fit is defined by the least squares method, which minimizes the sum of the squares of the residuals (errors).

Definition. 27.1: Residual

A residual, e_i, for the i-th data point is the difference between the actual value y_i and the predicted value $\hat{y}_i$:

$$e_i = y_i - \hat{y}_i = y_i - (\beta_0 + \beta_1 x_i)$$

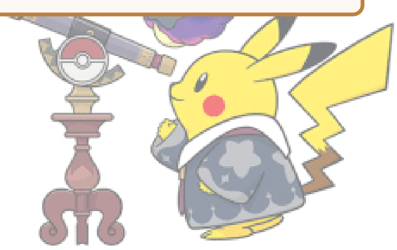
Definition. 27.2: Cost Function

The cost function, $J(\beta_0, \beta_1)$, is the quantity we want to minimize:

$$J(\beta_0, \beta_1) = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - (\beta_0 + \beta_1 x_i))^2$$

Theorem. 27.1: Linear Regression

To find the minimum of $J(\beta_0, \beta_1)$, we set the partial derivatives with respect to β_0 and β_1 to zero. This leads to the following solutions for the optimal coefficients, $\hat{\beta}_0$ and $\hat{\beta}_1$:



- The optimal slope coefficient is given by

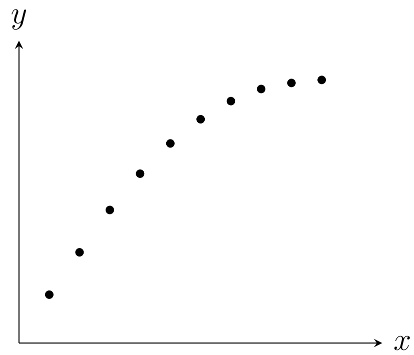
$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

- The optimal intercept coefficient is then found using the fact that the regression line must pass through the point of means $(\bar{x}, \bar{y})$:

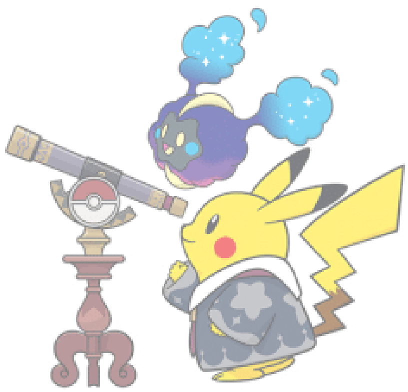
$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}$$

27.2 Nonlinear Regression

Nonlinear regression aims to fit a curve to data where the relationship between variables is **not linear**. While computers can do this accurately, it is possible to get an approximate fit manually using **eyes and pen**.



Draw a smooth curve that visually passes as close as possible to all points. Adjust the shape until it captures the overall trend.



Part III: Observational Astronomy

28 Instrumentation and Space Technologies

28.1 Telescope

28.1.1 Type of Telescope

Optical Telescope Refracting telescope uses lenses to bend (refract) light to a focal point.

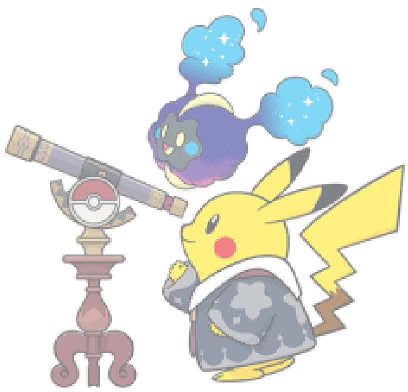
- **Advantages**: Sealed tube, no central obstruction, good for planetary observation
- **Disadvantages**: Chromatic aberration, size limitations, expensive
- **Structure**: Objective lens $\longrightarrow$ Tube $\longrightarrow$ Eyepiece

Reflecting telescope uses mirrors to reflect light to a focal point.

- **Types by Optical Design**:
  1. **Newtonian**: Primary parabolic mirror and flat secondary diagonal
  2. **Cassegrain**: Primary hyperbolic and secondary convex mirror
  3. **Ritchey-Chrétien**: Both mirrors hyperbolic (no spherical aberration)
  4. **Nasmyth**: Light directed to side of telescope
- **Advantages**: No chromatic aberration, easier to make large apertures
- **Disadvantages**: Central obstruction, coma in some designs

Radio Telescopes Radio telescopes collect radio waves using large parabolic dishes or arrays.

- **Examples**: Arecibo (305 m), FAST (500 m), ALMA (array)
- **Components**: Dish, feed antenna, receiver, amplifier, recorder



Mount Type	Advantages	Disadvantages
Altazimuth	Simple, compact	Field rotation
Equatorial	Natural tracking	Complex, heavy
Fork	Stable for Cassegrains	Limited sky access
German Equatorial	Good balance	Meridian flip required

Table 9: Comparison of telescope mount types

28.1.2 Mount Types

28.1.3 Key Components

1. **Primary Mirror/Objective**: Main light-gathering element
2. **Secondary Mirror**: Redirects light path (in reflectors)
3. **Focuser**: Adjusts position of eyepiece or instrument
4. **Eyepiece/Instrument**: Final light analysis
5. **Mount**: Supports and points the telescope
6. **Drive System**: Tracks celestial objects

28.1.4 Linear Magnification

The magnification of a telescope is the factor by which the telescope increases the apparent size of an object. It is given by

$$M = \frac{f_{\text{obj}}}{f_{\text{eye}}}$$

where f_{obj} is the focal length of the objective lens or mirror and f_{eye} is the focal length of the eyepiece.

28.1.5 Angular Magnification

Consider an object of height h placed at a distance D from the eye. The angle θ subtended at the eye (called the **visual angle**) is

$$\theta \approx \tan\theta = \frac{h}{D} \qquad (\theta \ll 1)$$

The approximation $\tan\theta \approx \theta$ is valid for small angles (measured in radians), which is typical in optical systems.

Angular magnification M is defined as

$$M = \frac{\theta'}{\theta}$$

where

- θ' is the angle subtended at the eye when viewing the object through the lens,
- θ is the angle subtended at the eye when viewing the object directly (unaided eye).

28.1.6 Chromatic Aberration

Chromatic aberration is an optical defect of lenses that arises due to the **dispersion** of light. It occurs because the refractive index of lens material depends on the wavelength of light.
The refractive index n of a transparent medium is a function of wavelength λ:

$$n = n(\lambda)$$

In general, shorter wavelengths (violet light) experience a higher refractive index than longer wavelengths (red light):

$$n_{\text{violet}} > n_{\text{red}}$$

The focal length f of a thin lens is given by the lens maker's formula (for thin lens)

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Since n depends on λ, the focal length also depends on wavelength:

$$f = f(\lambda)$$

Hence, different colors of light are brought to focus at different positions along the optical axis.

28.1.7 f-number

The focal ratio (or f-number) is the ratio of the telescope's focal length f to the diameter D of the aperture:

$$f/\# = \frac{f}{D}$$

A smaller focal ratio corresponds to a wider field of view and faster exposure times, which is important for observing faint objects.

28.1.8 Light-gathering Power

The light-gathering power of a telescope is its ability to collect light from an astronomical object. This is important because the more light a telescope can gather, the fainter objects it can detect. The light-gathering power is proportional to the area of the telescope's aperture, A, which is typically circular. The formula for the area of a circular aperture is

$$A = \pi \left(\frac{D}{2} \right)^2$$

where D is the diameter of the aperture. Therefore, the light-gathering power is proportional to the square of the aperture diameter:

$$L \propto D^2$$

This relationship means that a telescope with a larger aperture collects more light, allowing for observations of fainter objects.

28.1.9 Adaptive Optics

Adaptive optics is a technology used to improve the performance of optical systems by compensating for distortions caused by the Earth's atmosphere. These distortions, known as atmospheric turbulence, can blur images taken with ground-based telescopes. Adaptive optics systems use deformable mirrors to correct for these distortions in real-time. The process involves

1. A guide star or laser is used to create a reference point in the sky.
2. A wavefront sensor measures the distortion of the light coming from the guide star.
3. A computer calculates the necessary corrections to the wavefront.
4. A deformable mirror adjusts the light path, compensating for the distortion.

The result is sharper images with significantly reduced atmospheric distortion.

28.1.10 Artificial Light

Artificial light pollution affects astronomical observations, particularly in urban areas. The brightness of the night sky due to artificial lighting can drown out faint celestial objects, making it harder to observe stars, planets, and galaxies. Efforts such as **light pollution mitigation** are crucial for preserving the quality of astronomical observations.

28.2 Interferometer

Geometric Model of a Two-Element Interferometer A two-element interferometer uses two telescopes to observe the same astronomical object simultaneously. The signals from the two telescopes are combined, and the interference pattern is analyzed to extract higher resolution data than a single telescope could provide. For two telescopes separated by a distance D, the resolution of the interferometer is related to the wavelength λ of the observed light and the baseline D by the following formula for the angular resolution θ:

$$\theta \sim \frac{\lambda}{D}$$

Aperture Synthesis Aperture synthesis is a technique used in radio astronomy and interferometry to create an image with the resolution of an aperture much larger than the physical size of the telescope. The method involves observing the same object with multiple telescopes at different positions and combining the data to simulate the effect of a much larger telescope.

The resolution of a synthetic aperture is determined by the maximum separation between the telescopes, referred to as the baseline. By changing the positions of the telescopes, astronomers can collect data at multiple baselines, which allows them to improve the resolution over time.

28.3 Detector

28.3.1 Photometers

A photometer measures the intensity of light from a source in a specific wavelength band. It typically uses a single detector element and filters to isolate desired wavelengths.

28.3.2 Charge-Coupled Devices (CCDs)

CCDs are widely used digital detectors that convert photons into electrons and then into a measurable voltage. They offer high quantum efficiency and the ability to create 2D images. The number of electrons generated in a pixel can be calculated as

$$N_e = F_\gamma \, A \, t \, \mathrm{QE}$$

where

- F_γ is the photon flux (photons $s^{-1}m^{-2}$),
- A is the telescope collecting area (m^2),
- t is the exposure time (s),
- QE is the quantum efficiency of the detector.

28.4 Plate Scale

28.4.1 Introduction

For small angles (where $\tan\theta \approx \theta$), the plate scale P for telescope is given by

$$P = \frac{s}{\theta} = \frac{1}{f}$$

where

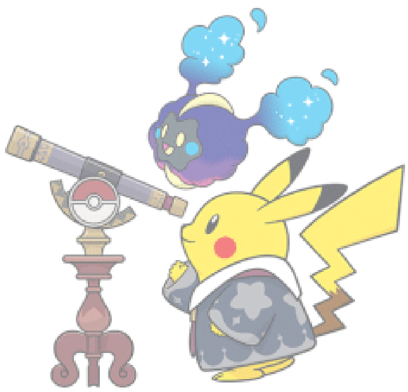
- θ is the angular size on sky (radians)
- s is the linear size in focal plane
- f is the telescope focal length

Since astronomers work with arcseconds and millimeters (or microns for pixels):

$$P_{\text{arcsec/mm}} = \frac{206265}{f_{\text{mm}}}$$

The constant 206,265 comes from:

$$1 \text{ radian} = \frac{180}{\pi} \times 3600 = 206264.8 \text{ arcseconds}$$



For digital detectors like CCDs,

$$P_{\text{arcsec/px}} = \frac{206265 \times p_{\mu m}}{f_{\text{mm}} \times 1000}$$

where $p_{\mu m}$ is the pixel size in micrometers.

28.4.2 Field of View

The total angular field covered by a detector is

$$\text{FOV}_{\text{width}} = N_x \cdot P_{\text{px}} \text{ [arcseconds]}$$

$$\text{FOV}_{\text{height}} = N_y \cdot P_{\text{px}} \text{ [arcseconds]}$$

where N_x and N_y are the number of pixels in each dimension.

28.5 Space-Based Instruments

Space-based instruments operate above Earth's atmosphere to observe the universe with high precision.

- **Atmospheric limitations**: Earth's atmosphere absorbs or distorts many wavelengths such as ultraviolet, X-ray, and infrared radiation.
- **Improved resolution**: absence of atmospheric turbulence allows sharper images and more stable measurements.
- **Instrument types**: telescopes, spectrometers, particle detectors, and magnetometers.
- **Constraints**: high cost, limited repair opportunities, and finite operational lifetimes.

28.6 Signal-to-Noise Ratio

28.6.1 Introduction

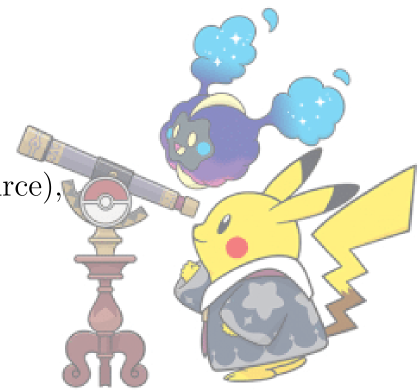
Signal-to-Noise Ratio (SNR) is a measure of the quality of a signal in the presence of noise. It quantifies how much the signal stands out from the background noise. High SNR means a clear signal, while low SNR indicates that noise dominates.

SNR is defined as

$$\text{SNR} = \frac{S}{N}$$

where

- S is the signal strength (e.g., number of detected photons from the source),
- N is the noise (e.g. standard deviation of the background).



28.6.2 Pure Poisson Noise

Definition. 28.1: Poisson Distribution

The Poisson distribution is a discrete probability distribution that expresses the probability of a number of events occurring in a fixed interval of time or space, given the average number of occurrences. It is commonly used to model random events like the number of phone calls received at a call center, the number of accidents at an intersection, or the number of particles decaying per unit time. The probability mass function of a Poisson distribution is given by

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \ldots$$

where

- X is the random variable representing the number of events,
- λ is the average number of occurrences in the given interval (also called the rate or intensity),
- $k!$ is the factorial of k, i.e., $k! = k(k-1)(k-2)\ldots 1$.

The Poisson distribution is characterized by the mean and variance both being equal to λ, i.e.,

$$\mu = \sigma^2 = \lambda$$

For photon-counting detectors, the dominant noise is often Poisson. If the expected signal is S photons, then Poisson statistics give

$$\sigma = \sqrt{S}$$

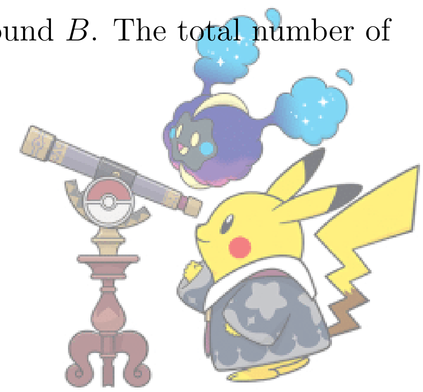
Hence,

$$\boxed{\text{SNR} = \frac{S}{\sqrt{S}} = \sqrt{S}}$$

28.6.3 Background Noise

Suppose the measurement includes both the object signal S and the background B. The total number of detected photons is

$$S_{\text{tot}} = S + B$$



Poisson statistics give total variance

$$\sigma^2 = S + B$$

Hence,

$$\boxed{\text{SNR} = \frac{S}{\sqrt{S + B}}}$$

28.6.4 Readout Noise

A CCD or CMOS detector adds readout noise σ_{read} per pixel. If n_{pix} pixels are used, the read noise contribution is

$$\sigma^2_{\text{read,tot}} = n_{\text{pix}} \sigma^2_{\text{read}}$$

The total variance becomes

$$\sigma^2 = S + B + n_{\text{pix}} \sigma^2_{\text{read}}$$

Hence,

$$\boxed{\text{SNR} = \frac{S}{\sqrt{S + B + n_{\text{pix}} \sigma^2_{\text{read}}}}}$$

28.6.5 Complete Equation

$$\boxed{\text{SNR} = \frac{S}{\sqrt{S + B + n_{\text{pix}} \left(1 + \frac{n_{\text{pix}}}{n_B}\right) (N_s + N_D + \sigma^2_{\text{read}} + G^2 \sigma_f^2)}}}$$

where

- n_B is the background pixels in the image, which correspond to regions with no object or light source.
- N_s represents the signal from the source, representing the number of photons from the object being observed.
- N_D represents the dark noise, which arises from thermally generated electrons in the CCD detector, even in the absence of light.
- G is the gain of the CCD detector, which relates the number of electrons collected by the pixel to the digital output recorded by the system.
- σ_f is the fluctuations or Fano noise, which arises from the statistical nature of electron counting processes.